Section 3.5 - Basic Differentiation Properties

Objectives:

- The student will be able to calculate the derivative of a constant function.
- The student will be able to apply the power rule.
- The student will be able to apply the constant multiple and sum and difference properties.
- The student will be able to solve applications.

Derivative Notation

In the preceding section we defined the derivative of a function. There are several widely used symbols to represent the derivative. Given \( y = f(x) \), the derivative of \( f \) at \( x \) may be represented by any of the following:

- \( f'(x) \)
- \( y' \)
- \( \frac{dy}{dx} \)

We will see some nice rules that allow us to quickly find the derivative of a function without using the definition of derivative every time.

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Ex: What is the slope of a constant function?

The graph of \( f(x) = C \) is a horizontal line with slope 0, so we would expect \( f'(x) = 0 \).
Theorem 1. Let $y = f(x) = b$ be a constant function, then $y' = f'(x) = 0$. The derivative of a constant is zero.

Ex. $f(x) = 3$ then $f'(x) = 0$

Power Rule
A function of the form $f(x) = x^n$ is called a power function. This includes $f(x) = x$ (where $n = 1$) and radical functions (fractional $n$).

Theorem 2. (Power Rule) Let $n$ be an integer and $f(x) = x^n$ be a power function, then

$$y' = f'(x) = \frac{dy}{dx} = n(x^{n-1})$$

**THEOREM 2 IS VERY IMPORTANT. IT WILL BE USED A LOT!**

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
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</thead>
<tbody>
<tr>
<td>$f(x) = x^2$</td>
<td>$f'(x) = 2x^{2-1} = 2x = 2x$</td>
</tr>
<tr>
<td>$f(x) = x^3$</td>
<td>$f'(x) = 3x^{3-1} = 3x^2$</td>
</tr>
<tr>
<td>$f(x) = x$</td>
<td>$f'(x) = 1x^{1-1} = 1x^0 = 1$</td>
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<tr>
<td>$f(x) = x^{-2}$</td>
<td>$f'(x) = -2x^{-2-1} = -2x^{-3} = -2\frac{x^{-3}}{x^2}$</td>
</tr>
<tr>
<td>$f(x) = \frac{1}{x^3} = x^{-3}$</td>
<td>$f'(x) = -3x^{-3-1} = -3x^{-4} = -3\frac{x^{-4}}{x^2}$</td>
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Ex. Differentiate $f(x) = x^5$. 
Ex: Differentiate $f(x) = \sqrt{x}$.

We can extend our power rule to rational exponents.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$$

Constant Multiple Property

**Theorem 3.** Let $y = f(x) = k \cdot u(x)$ be a constant $k$ times a function $u(x)$. Then

$$y' = f'(x) = k \cdot u'(x).$$

In words: The derivative of a constant times a function is the constant times the derivative of the function.

Ex: Differentiate $f(x) = 7x^4$. 
Examples:
Differentiate $f(x) = \frac{2}{x}$

Differentiate $f(x) = \frac{4x^2}{5}$

Differentiate $f(x) = \frac{-3x}{2}$

Differentiate $f(x) = 2\sqrt{x}$
Differentiate \( f(x) = \frac{1}{2\sqrt[3]{x^2}} \)

Sum and Difference Properties

**Theorem 5.** If \( y = f(x) = u(x) \pm v(x) \), then
\[
y' = f'(x) = u'(x) \pm v'(x).
\]

**In words:**

- The derivative of the **sum** of two differentiable functions is the sum of the derivatives.
- The derivative of the **difference** of two differentiable functions is the difference of the derivatives.

\[
\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} [f(x)] \pm \frac{d}{dx} [g(x)]
\]

The derivative of a sum or difference is the sum or difference of the derivatives

***Caution*** It is **NOT** true that derivatives...

\[
\frac{d}{dx} [f(x) \cdot g(x)] \neq \frac{d}{dx} [f(x)] \cdot \frac{d}{dx} [g(x)]
\]
Ex. Differentiate $f(x) = [x^3 - 4x + 5]$

Ex. Differentiate $f(x) = \left[\frac{-x^4}{2} + 3x^3 - 2x\right]$

Applications
Remember that the derivative gives the **instantaneous rate of change** of the function with respect to $x$. That might be:

- Instantaneous velocity.
- Tangent line slope at a point on the curve of the function.
- Marginal Cost. If $C(x)$ is the cost function, that is, the total cost of producing $x$ items, then $C'(x)$ approximates the cost of producing one more item at a production level of $x$ items. $C'(x)$ is called the **marginal cost**.

Application1: $f'(c)$ is the slope of the tangent line to the graph of $f$ at $x=c$
Application2: If $y=f(x)$, then $f'(c)$ is the rate of change of $y$ with respect to $x$ at $x=c$. 
A common use of the rate of change is to describe the motion of an object moving in a straight line. The **position function** is denoted by $s(t)$ and is defined as $s(t) = \text{position of object at time } t$

In general, the velocity of the object at time $t$ is given by.

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = s'(t)$$

Velocity can be negative, zero or positive. The speed of the object is the absolute value of its velocity. Speed cannot be negative.

Ex: The position of a free-falling object under the influence of gravity can be modeled by the position function. $s(t) = -16t^2 + v_0t + s_0$, where $v_0$ is the initial velocity of the object, $s_0$, is the initial position measured in feet, and time is measured in seconds.

A billiard ball is dropped from a height of 100 feet. Its position function is given by $s(t) = -16t^2 + 100$

a. When does the ball hit the ground?
b. What is the ball’s velocity at impact?

Tangent Line Example
Let $f(x) = x^4 - 6x^2 + 10$.
(a) Find $f'(x)$
(b) Find the equation of the tangent line at $x = 1$
Application Example
The total cost (in dollars) of producing $x$ portable radios per day is

$$C(x) = 1000 + 100x - 0.5x^2 \quad \text{for} \quad 0 \leq x \leq 100.$$  

1. Find the marginal cost at a production level of $x$ radios.

2. Find the marginal cost at a production level of 80 radios and interpret the result.

3. Find the actual cost of producing the 81st radio and compare this with the marginal cost.

Summary
- If $f(x) = C$, then $f'(x) = 0$
- If $f(x) = x^n$, then $f'(x) = n x^{n-1}$
- If $f(x) = k \cdot u(x)$, then $f'(x) = k \cdot u'(x)$
- If $f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$. 
Section 3.6 - Differentials

Objectives:
- The student will be able to apply the concept of increments.
- The student will be able to compute differentials.
- The student will be able to calculate approximations using differentials.

Increments
In a previous section we defined the derivative of \( f \) at \( x \) as the limit of the difference quotient:

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

Increment notation will enable us to interpret the numerator and the denominator of the difference quotient separately.

Ex: Let \( y = f(x) = x^3 \).

If \( x \) changes from 2 to 2.1, then \( y \) will change from \( y = f(2) = 8 \) to \( y = f(2.1) = 9.261 \).

We can write this using increment notation. The change in \( x \) is called the increment in \( x \) and is denoted by \( \Delta x \). \( \Delta \) is the Greek letter “delta”, which often stands for a difference or change. Similarly, the change in \( y \) is called the increment in \( y \) and is denoted by \( \Delta y \).

In our example,

\[
\Delta x = 2.1 - 2 = 0.1 \\
\Delta y = f(2.1) - f(2) = 9.261 - 8 = 1.261.
\]
Graphical Illustration of Increments

For \( y = f(x) \)

\[
\Delta x = x_2 - x_1 \quad \Delta y = y_2 - y_1
\]

\( x_2 = x_1 + \Delta x = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1) \)

- \( \Delta y \) represents the change in \( y \) corresponding to a \( \Delta x \) change in \( x \).
- \( \Delta x \) can be either positive or negative.

\[ \Delta y \text{ represents the change in } y \text{ corresponding to a } \Delta x \text{ change in } x. \]

\[ \Delta x \text{ can be either positive or negative.} \]

![Graphical Illustration](image)

Differentials

Assume that the limit

\[ f''(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

exists.

For small \( \Delta x \),

\[ f'(x) \approx \frac{\Delta y}{\Delta x} \]

Multiplying both sides of this equation by \( \Delta x \) gives us

\[ \Delta y \approx f'(x) \Delta x. \]

Here the increments \( \Delta x \) and \( \Delta y \) represent the actual changes in \( x \) and \( y \).

One of the notations for the derivative is

\[ f'(x) = \frac{dy}{dx} \]

If we pretend that \( dx \) and \( dy \) are actual quantities, we get

\[ dy = f'(x) \, dx \]

We treat this equation as a definition, and call \( dx \) and \( dy \) differentials.
Interpretation of Differentials

$\Delta x$ and $dx$ are the same, and represent the change in $x$.

The increment $\Delta y$ stands for the **actual change** in $y$ resulting from the change in $x$.

The differential $dy$ stands for the **approximate change** in $y$, estimated by using derivatives. $\Delta y \approx dy = f'(x) \, dx$

In applications, we use $dy$ (which is easy to calculate) to estimate $\Delta y$ (which is what we want).

Ex: Find $dy$ for $f(x) = x^2 + 3x$ and evaluate $dy$ for $x = 2$ and $dx = 0.1$. 
Ex: Cost-Revenue
A company manufactures and sells $x$ transistor radios per week. If the weekly cost and revenue equations are

\[
C(x) = 5000 + 2x \\
R(x) = 10x - \frac{x^2}{1000}
\]

find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 units/week.
Section 3.7 - Marginal Analysis in Business and Economics

Objectives:
The student will be able to compute:
- Marginal cost, revenue and profit
- Marginal average cost, revenue and profit
- The student will be able to solve applications

Marginal Cost
Remember that marginal refers to an instantaneous rate of change, that is, a derivative.

**Definition:**
If $x$ is the number of units of a product produced in some time interval, then
\[
\text{Total cost} = C(x) \\
\textbf{Marginal cost} = C'(x)
\]

Marginal Revenue and Marginal Profit

**Definition:**
If $x$ is the number of units of a product sold in some time interval, then
\[
\text{Total revenue} = R(x) \\
\textbf{Marginal revenue} = R'(x)
\]

If $x$ is the number of units of a product produced and sold in some time interval, then
\[
\text{Total profit} = P(x) = R(x) - C(x) \\
\textbf{Marginal profit} = P'(x) = R'(x) - C'(x)
\]
Marginal Cost and Exact Cost
Assume $C(x)$ is the total cost of producing $x$ items. Then the exact cost of producing the $(x + 1)$st item is
\[ C(x + 1) - C(x). \]
The marginal cost is an approximation of the exact cost.
\[ C'(x) \approx C(x + 1) - C(x). \]
Similar statements are true for revenue and profit.

Ex: The total cost of producing $x$ electric guitars is
\[ C(x) = 1,000 + 100x - 0.25x^2. \]

1. Find the exact cost of producing the 51st guitar.

2. Use the marginal cost to approximate the cost of producing the 51st guitar.
Marginal Average Cost

**Definition:** If $x$ is the number of units of a product produced in some time interval, then

Average cost per unit = \[ \bar{C}(x) = \frac{C(x)}{x} \]

**Marginal average cost** = \[ \bar{C}'(x) = \frac{d}{dx} \bar{C}(x) \]

Marginal Average Revenue

If $x$ is the number of units of a product sold in some time interval, then

Average revenue per unit = \[ \bar{R}(x) = \frac{R(x)}{x} \]

**Marginal average revenue** = \[ \bar{R}'(x) = \frac{d}{dx} \bar{R}(x) \]

Marginal Average Profit

If $x$ is the number of units of a product produced and sold in some time interval, then

Average profit per unit = \[ \bar{P}(x) = \frac{P(x)}{x} \]

**Marginal average profit** = \[ \bar{P}'(x) = \frac{d}{dx} \bar{P}(x) \]

Warning! To calculate the marginal averages you must calculate the average first (divide by $x$), and then the derivative. If you change this order you will get no useful economic interpretations.
Ex: The total cost of printing $x$ dictionaries is $C(x) = 20,000 + 10x$

1. Find the average cost per unit if 1,000 dictionaries are produced.

2. Find the marginal average cost at a production level of 1,000 dictionaries, and interpret the results.

3. Use the results from above to estimate the average cost per dictionary if 1,001 dictionaries are produced.
Ex: The price-demand equation and the cost function for the production of television sets are given by

\[ p(x) = 300 - \frac{x}{30} \quad \text{and} \quad C(x) = 150,000 + 30x \]

where \( x \) is the number of sets that can be sold at a price of $p per set, and \( C(x) \) is the total cost of producing \( x \) sets.

1. Find the marginal cost.

2. Find the revenue function in terms of \( x \).

3. Find the marginal revenue.

4. Find \( R'(1500) \) and interpret the results.
5. Graph the cost function and the revenue function on the same coordinate. Find the break-even point. $0 \leq x \leq 9,000$ and $0 \leq y \leq 700,000$

6. Find the profit function in terms of $x$.

7. Find the marginal profit.

8. Find $P'(1500)$ and interpret the results.