Notations for Matrices

We can represent a matrix in two different ways.
1. A capital letter, such as A, B, or C, can denote a matrix.
2. A lowercase letter enclosed in brackets, such as that shown below can denote a matrix.
   \[ A = \begin{bmatrix} a_{ij} \end{bmatrix} \]

A general element in matrix A is denoted by \( a_{ij} \). This refers to the element in the \( i^{th} \) row and \( j^{th} \) column. \( a_{32} \) is the element located in the 3rd row, 2nd column.

See below.

A matrix of order \( m \times n \) has \( m \) rows and \( n \) columns. If \( m=n \), a matrix has the same number of rows as columns and is called a square matrix.

Example -

Let \( A = \begin{bmatrix} 1 & -3 & 3 \\ -4 & -5 & -3 \\ -3 & -2 & 4 \end{bmatrix} \)

a. What is the order of A?

b. If \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \), identify \( a_{23} \), and \( a_{31} \)
Equality of Matrices

**Definition of Equality of Matrices**

Two matrices \( A \) and \( B \) are equal if and only if they have the same order \( m \times n \) and \( a_{ij} = b_{ij} \) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

For example, if \( A = \begin{bmatrix} x & y + 1 \\ z & 6 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} \), then \( A = B \) if and only if \( x = 1 \), \( y + 1 = 5 \) (so \( y = 4 \)), and \( z = 3 \).

Matrix Addition and Subtraction

Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be matrices of order \( m \times n \).

<table>
<thead>
<tr>
<th>Definition</th>
<th>The Definition in Words</th>
<th>Example</th>
</tr>
</thead>
</table>
| **Matrix Addition**         | Matrices of the same order are added by adding the elements in corresponding positions. | \[
\begin{bmatrix}
1 & -2 \\
3 & 5
\end{bmatrix} + \begin{bmatrix}
-1 & 6 \\
3 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 4 \\
3 & 9
\end{bmatrix}
\]                           |
| **Matrix Subtraction**      | Matrices of the same order are subtracted by subtracting the elements in corresponding positions. | \[
\begin{bmatrix}
1 & -2 \\
3 & 5
\end{bmatrix} - \begin{bmatrix}
-1 & 6 \\
3 & 0
\end{bmatrix} = \begin{bmatrix}
2 & -8 \\
3 & 1
\end{bmatrix}
\]                           |

Properties of Matrix Addition

If \( A, B, \) and \( C \) are \( m \times n \) matrices and 0 is the \( m \times n \) zero matrix, then the following properties are true.

1. \( A + B = B + A \) \hspace{1cm} Commutative Property of Addition
2. \((A + B) + C = A + (B + C)\) \hspace{1cm} Associative Property of Addition
3. \( A + 0 = 0 + A = A \) \hspace{1cm} Additive Identity Property
4. \( A + (-A) = (-A) + A = 0 \) \hspace{1cm} Additive Inverse Property

Zero matrix = \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Additive Inverses: The inverse of \[
\begin{bmatrix}
-3 & 2 \\
1 & 5
\end{bmatrix}
\] is \[
\begin{bmatrix}
3 & -2 \\
-1 & -5
\end{bmatrix}
\] .

\[
\begin{bmatrix}
-3 & 2 \\
1 & 5
\end{bmatrix} + \begin{bmatrix}
3 & -2 \\
-1 & -5
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

If \( A = \begin{bmatrix}
-3 & 2 \\
1 & 5
\end{bmatrix} \) then \( -A = \begin{bmatrix}
3 & -2 \\
-1 & -5
\end{bmatrix} \)
Example - Subtract the following two matrices.

\[
\begin{pmatrix}
3 & -1 \\
0 & 4
\end{pmatrix}
- \begin{pmatrix}
2 & 1 \\
-4 & 3
\end{pmatrix}
\]

Example - What is the zero matrix for all $2 \times 3$ matrices?

What is the additive inverse for the matrix below?

\[
\begin{pmatrix}
3 & -1 \\
0 & 4
\end{pmatrix}
\]
Scalar Multiplication

Definition of Scalar Multiplication
If $A = [a_{ij}]$ is a matrix of order $m \times n$ and $c$ is a scalar, then the matrix $cA$ is the $m \times n$ matrix given by $cA = [ca_{ij}]$.

This matrix is obtained by multiplying each element of $A$ by the real number $c$. We call $cA$ a scalar multiple of $A$.

Example - If $A = \begin{pmatrix} 2 & 5 \\ -3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 4 & 0 \end{pmatrix}$, what is $2A + 3B$?

Properties of Scalar Multiplication
If $A$ and $B$ are $m \times n$ matrices, and $c$ and $d$ are scalars, then the following properties are true.

1. $(cd)A = c(dA)$  
   Associative Property of Scalar Multiplication
2. $1A = A$  
   Scalar Identity Property
3. $c(A + B) = cA + cB$  
   Distributive Property
4. $(c + d)A = cA + dA$  
   Distributive Property
Matrix Multiplication

Definition of Matrix Multiplication: $2 \times 2$ Matrices

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

1. Multiply each element in row 1 of $A$ by the corresponding element in column 1 of $B$.
2. Add these products.
3. Record the sum as the element in row 1, column 1 of the product matrix.

Study Tip

Writing the location of each element in the product matrix $AB$ may help you to remember how to multiply $2 \times 2$ matrices.

$$AB = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Row 1 (of $A$) \times Column 1 (of $B$) Row 1 \times Column 2

Row 2 \times Column 1 Row 2 \times Column 2
Definition of Matrix Multiplication

The product of an \( m \times n \) matrix, \( A \), and an \( n \times p \) matrix, \( B \), is an \( m \times p \) matrix, \( AB \), whose elements are found as follows. The element in the \( i \)th row and \( j \)th column of \( AB \) is found by multiplying each element in the \( i \)th row of \( A \) by the corresponding element in the \( j \)th column of \( B \) and adding the products.

Notice that when multiplying two matrices the order of the product matrix is the order of the two outside dimensions from the original matrices. Because matrix multiplication is not commutative be careful about the order in which matrices appear when performing this operation.

\[
AB = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 32 \\ 6 \end{bmatrix}
\]

\[
BA = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 10 & 15 & 18 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}
\]
Properties of Matrix Multiplication

If $A$, $B$, and $C$ are matrices and $c$ is a scalar, then the following properties are true. (Assume the order of each matrix is such that all operations in these properties are defined.)

1. $(AB)C = A(BC)$  \hspace{1cm} \text{Associative Property of Matrix Multiplication}
2. $A(B + C) = AB + AC$  \hspace{1cm} \text{Distributive Properties of Matrix Multiplication}
   \hspace{1cm} (A + B)C = AC + BC
3. $c(AB) = (cA)B$  \hspace{1cm} \text{Associative Property of Scalar Multiplication}

Find the product.

Example -

$$
\begin{bmatrix}
-2 & 1 & 3 \\
1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
-4 \\
1 \\
2
\end{bmatrix}
$$
Example -

\[
\begin{bmatrix}
-1 & 2 \\
0 & 1 \\
3 & -1 \\
1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
4 & 0 \\
-2 & -3 \\
\end{bmatrix}
\]