Pre-Calculus I

3.5 – Exponential Growth and Decay

Exponential Growth and Decay Models

\[ f(t) = A_0 e^{kt} \quad \text{or} \quad A = A_0 e^{kt} \]

- If \( k > 0 \), the function models the amount, or size, of a growing entity. \( A_0 \) is the original amount, or size, of the growing entity at time \( t = 0 \), \( A \) is the amount at time \( t \), and \( k \) is a constant representing the growth time.

- If \( k < 0 \), the function models the amount, or size, of a decaying entity. \( A_0 \) is the original amount, or size, of the decaying entity at time \( t = 0 \), \( A \) is the amount at time \( t \), and \( k \) is a constant representing the decay rate.

Example – The below graph shows the US population in millions for five selected years from 1970 to 2007.


- 1970: 203.3
- 1980: 226.5
- 1990: 248.7
- 2000: 281.4
- 2007: 300.9
1. Find an exponential growth function that models the data for 1970 through 2007.

2. By which year will the US population reach 312 million?

3. Is this example an exponential growth or decay?
Example: Carbon-14 decays exponentially with a half-life of approximately 5715 years. The **half-life** of a substance is the time required for half of a given sample to disintegrate. Thus after 5715 years a given amount of carbon-14 will have decayed to half the original amount

1. Find an exponential decay model for carbon-14

2. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls.
Logistic Growth Models: The mathematical model for limited logistic growth is given by \( f(t) = \frac{c}{1 + ae^{-bt}} \) or \( A = \frac{c}{1 + ae^{-bt}} \), where \( a, b, \text{ and } c \) are constants with \( c > 0 \) and \( b > 0 \). Note that as time increases to infinity then the expression goes to 0.

Example – The following models describes the number of people, \( f(t) \), who have become ill with influenza \( t \) weeks after its initial outbreak in a town with 30,000 inhabitants.

\[
f(t) = \frac{30,000}{1 + 20e^{-1.5t}}
\]

1. How many people became ill with the flu when the epidemic began? \( (t=0) \)

2. How many people were ill by the end of the forth week? \( (t=4) \)

3. What is the limiting size of \( f(t) \), the population that became ill?
Newton’s Law of Cooling – The temperature $T$, of a heated object at time $t$ is given by $T = C + (T_0 - C)e^{kt}$, where $C$ is the constant temperature of the surrounding medium, $T_0$ is the initial temperature of the heated object, and $k$ is a negative constant that is associated with the cooling object.

Example: A cake removed from the oven has a temperature of 210°F. It is left to cool in a room that has a temperature of 70°F. After 30 minutes, the temperature of the cake is 140°F.

1. Use Newton’s Law of Cooling to find a model for the temperature of the cake, $T$, after $t$ minutes.
2. What is the temperature of the cake after 40 minutes?

3. When will the temperature of the cake be 90°F?
Choosing a Model for Data

If the graph of your data increases/ decreases rapidly then begins to level off a bit then choose a logarithmic model

If the graph of your data increases/ decreases more and more rapidly then choose an exponential model

Example: Choose either Logarithmic or Exponential model

Number of Weight Loss Surgeries in The US
Expressing $y = ab^x$ in base $e$

$y = ab^x$ is equivalent to $y = ae^{(lnb)x}$

Example – Rewrite the following

1. $g(x) = 2.569(1.017)^x$

2. $y = 4(7.8)^x$

3. $h(x) = 4.5(0.6)^x$