Chapter 12 – Probability

Section 12.1

Probability

The study of probability is concerned with random phenomena. Even though we cannot be certain whether a given result will occur, we often can obtain a good measure of its likelihood, or probability.

Definitions

- An **experiment** is a controlled operation that yields a set of results.
- The possible results of an experiment are called its **outcomes**.
- An **event** is a subcollection of the outcomes of an experiment.
- **Empirical probability** is the relative frequency of occurrence of an event and is determined by actual observations of an experiment.
- **Theoretical probability** is determined through a study of the possible outcomes that can occur for the given experiment.

Theoretical Probability Formula

If all outcomes in a sample space $S$ are equally likely, and $E$ is an event within that sample space, then the **theoretical probability** of the event $E$ is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$  

**Example: Flipping a Cup**

A cup is flipped 100 times. It lands on its side 84 times, on its bottom 6 times, and on its top 10 times. Find the probability that it will land on its top.

Empirical Probability Formula

If $E$ is an event that may happen when an experiment is performed, then the **empirical probability** of event $E$ is given by

$$P(E) \approx \frac{\text{number of times event } E \text{ occurred}}{\text{number of times the experiment was performed}}.$$
**Example: Card Hands**
There are 2,598,960 possible hands in poker. If there are 36 possible ways to have a straight flush, find the probability of being dealt a straight flush.

**Example: Gender of a Student**
A school has 820 male students and 835 female students. If a student from the school is selected at random, what is the probability that the student would be a female?

**Example** - In 100 tosses of a fair die, 19 landed showing a 3. Find the empirical probability of the die landing showing a 3.

The Law of Large Numbers
- The **law of large numbers** states that probability statements apply in practice to a large number of trials, not to a single trial. It is the relative frequency over the long run that is accurately predictable, not individual events or precise totals.

As an experiment is repeated more and more times, the proportion of outcomes favorable to any particular event will tend to come closer and closer to the theoretical probability of that event.
Section 12.2 - Theoretical Probability

If all outcomes in a sample space $S$ are equally likely, and $E$ is an event within that sample space, then the theoretical probability of the event $E$ is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$ 

Equally likely outcomes

- If each outcome of an experiment has the same chance of occurring as any other outcome, they are said to be equally likely outcomes.
- For equally likely outcomes, the probability of Event $E$ may be calculated with the following formula.

**Example** - A die is rolled. Find the probability of rolling

a) a 2.
b) an odd number.
c) a number less than 4.
d) an 8.
e) a number less than 9.
Important Facts

- The probability of an event that cannot occur is 0.
- The probability of an event that must occur is 1.
- Every probability is a number between 0 and 1 inclusive; that is, $0 \leq P(E) \leq 1$.
- The sum of the probabilities of all possible outcomes of an experiment is 1.

Example - A standard deck of cards is well shuffled. Find the probability that the card is selected.

a) a 10.
b) not a 10.
c) a heart.
d) an ace, 2, or 3.
e) diamond and spade.
f) a card greater than 4 and less than 7.
Section 12.3 – Odds

**Odds** compare the number of favorable outcomes to the number of unfavorable outcomes. Odds are commonly quoted in horse racing, lotteries, and most other gambling situations.

If all outcomes in a sample space are equally likely, \( a \) of them are favorable to the event \( E \), and the remaining \( b \) outcomes are unfavorable to \( E \), then the **odds in favor** of \( E \) are \( a \) to \( b \), and the **odds against** \( E \) are \( b \) to \( a \).

\[
\text{Odds against event} = \frac{P(\text{event fails to occur})}{P(\text{event occurs})} = \frac{P(\text{failure})}{P(\text{success})}
\]

**Example** - Find the odds against rolling a 5 on one roll of a die.

\[
\text{Odds in favor of event} = \frac{P(\text{event occurs})}{P(\text{event fails to occur})} = \frac{P(\text{success})}{P(\text{failure})}
\]

**Example** - Find the odds in favor of landing on blue in one spin of the spinner.
Example - The odds against spinning a blue on a certain spinner are 4:3. Find the probability that

a) a blue is spun.
b) a blue is not spun.
Section 12.4 – Expected Value

Expected Value

Children in third grade were surveyed and told to pick the number of hours that they play electronic games each day. The probability distribution is given below.

<table>
<thead>
<tr>
<th># of Hours $x$</th>
<th>Probability $P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>.4</td>
</tr>
<tr>
<td>2</td>
<td>.2</td>
</tr>
<tr>
<td>3</td>
<td>.1</td>
</tr>
</tbody>
</table>

Compute a “weighted average” by multiplying each possible time value by its probability and then adding the products.

$$0(.3) + 1(.4) + 2(.2) + 3(.1) = 1.1$$

1.1 hours is the expected value (or the mathematical expectation) of the quantity of time spent playing electronic games.

If a random variable $x$ can have any of the values $x_1$, $x_2$, $x_3$, ..., $x_n$, and the corresponding probabilities of these values occurring are $P(x_1)$, $P(x_2)$, $P(x_3)$, ..., $P(x_n)$, then the expected value of $x$ is given by

$$E(x) = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + \cdots + x_n \cdot P(x_n).$$

The symbol $P_1$ represents the probability that the first event will occur, and $A_1$ represents the net amount won or lost if the first event occurs.

$P_2$ is the probability of the second event, and $A_2$ is the net amount won or lost if the second event occurs.

And so on...
Example - Teresa is taking a multiple-choice test in which there are four possible answers for each question. The instructor indicated that she will be awarded 3 points for each correct answer and she will lose 1 point for each incorrect answer and no points will be awarded or subtracted for answers left blank.

a) If Teresa does not know the correct answer to a question, is it to her advantage or disadvantage to guess?
b) If she can eliminate one of the possible choices, is it to her advantage or disadvantage to guess at the answer?

Example - When Calvin Winters attends a tree farm event, he is given a free ticket for the $75 door prize. A total of 150 tickets will be given out. Determine his expectation of winning the door prize.
**Example** - When Calvin Winters attends a tree farm event, he is given the opportunity to purchase a ticket for the $75 door prize. The cost of the ticket is $3, and 150 tickets will be sold. Determine Calvin’s expectation if he purchases one ticket.

**Games and Gambling**

A game in which the expected net winnings are zero is called a **fair game**. A game with negative expected winnings is unfair against the player. A game with positive expected net winnings is unfair in favor of the player.

Fair price = expected value + cost to play
Example - Suppose you are playing a game in which you spin the pointer shown in the figure, and you are awarded the amount shown under the pointer. If it costs $10 to play the game, determine:

a) the expectation of the person who plays the game.
b) the fair price to play the game.
Section 12.5 – Tree Diagrams

Uniformity and the Fundamental Counting Principal – If all we need is the total number of possibilities then an actual listing usually is unnecessary and often is difficult or tedious to obtain, especially if the list is long.
Below is the tree diagram from all non-repeating three-digit numbers with digits from the set \{1, 2, 3\}

The tree diagram is “uniform” in the sense that a given part of the task can be done in the same number of ways no matter which choices were selected for previous parts. For example, there are always two choices for the second digit (if the first digit is 1 then the second can be 2 or 3, if the first digit is 2 then the second can be 1 or 2, if the first digit is 3 then the second can be 1 or 2)

Part (a) of this problem was three digit numbers that can be written using digits from the set \{1, 2, 3\}, assuming that repeated digits are allowed. We found that there are a lot more possibilities in that case, there were 27 as opposed to 6, but the idea of “uniform” is still there because no matter what the first digit is, there are three choices for the second and three choices for third \{1, 2, 3\}

Counting Principle
If a first experiment can be performed in \(M\) distinct ways and a second experiment can be performed in \(N\) distinct ways, then the two experiments in that specific order can be performed in \(M \cdot N\) distinct ways
Definitions

- Sample space: A list of all possible outcomes of an experiment.
- Sample point: Each individual outcome in the sample space.
- Tree diagrams are helpful in determining sample spaces.

Example - Two balls are to be selected without replacement from a bag that contains one purple, one blue, and one green ball.

a) Use the counting principle to determine the number of points in the sample space.
b) Construct a tree diagram and list the sample space.
c) Find the probability that one blue ball is selected.
d) Find the probability that a purple ball followed by a green is selected.

\[
P(\text{event happening at least once}) = 1 - P(\text{event does not happen})
\]
Section 12.8 - The Counting Principle and Permutations

Counting Principle
If a first experiment can be performed in $M$ distinct ways and a second experiment can be performed in $N$ distinct ways, then the two experiments in that specific order can be performed in $M \cdot N$ distinct ways.

Example - A password to logon to a computer system is to consist of 3 letters followed by 3 digits. Determine how many different passwords are possible if:

a) repetition of letters and digits is permitted
b) repetition of letters and digits is not permitted
c) the first letter must be a vowel (a, e, i, o, u), the first digit cannot be 0, and repetition of letters and digits is not permitted.
Permutations

A permutation is any ordered arrangement of a given set of objects.

Permutations – arrangements are often called permutations, the number of permutations of \(n\) distinct things taken \(r\) at a time is denoted \(n^P_r\). Since the number of objects being arranged cannot exceed the total number available, we assume for our purposes here that \(r \leq n\). Applying the fundamental counting principle to arrangements of this type gives:

\[
\begin{align*}
    n^P_r &= n(n - 1)(n - 2) \ldots [n - (r - 1)] \\
    &= n(n - 1)(n - 2) \ldots [n - r + 1] \\
    &= n(n - 1)(n - 2) \ldots [n - r + 1] \\
    &= \frac{n(n - 1)(n - 2) \ldots (n - r + 1)(n - r)}{(n - r)(n - r - 1) \ldots 2 \times 1} \\
    &= \frac{n!}{(n-r)!}
\end{align*}
\]

Permutations are to evaluate the number of arrangements of \(n\) things taken \(r\) at a time, where repetitions are not allowed, and the order of the items is important.

Factorial Formula for Permutations

The number of permutations, or arrangements, of \(n\) distinct things taken \(r\) at a time, where \(r \leq n\), can be calculated as:

\[
n^P_r = \frac{n!}{(n-r)!}
\]

Alternative Notations are \(P(n,r)\) and \(P^n_r\).

For Example, \(4^P_2\) means “the number of permutations of 4 distinct things taken 2 at a time”.

Example - How many ways can 6 different stuffed animals be arranged in a line on a shelf?
Example - Consider the six numbers 1, 2, 3, 4, 5 and 6. In how many distinct ways can three numbers be selected and arranged if repetition is not allowed?

Example - The swimming coach has 8 swimmers who can compete in a “new” 100m relay (butterfly, backstroke, free style), he must select 3 swimmers, one for each leg of the relay in the event. In how many ways could he select the 3 swimmers?

Permutations of Duplicate Objects

The number of distinct permutations of \( n \) objects where \( n_1 \) of the objects are identical, \( n_2 \) of the objects are identical, ..., \( n_r \) of the objects are identical is found by the formula

\[
\frac{n!}{n_1!n_2! \cdots n_r!}
\]

Example - In how many different ways can the letters of the word “CINCINNATI” be arranged?

Evaluate each permutation

Using a graphing calculator we can perform this calculation directly as follows for a TI-83:

For \( 10P_6 \) enter in 10 then hit MATH – Scroll over to PRB and scroll down to 2 (nPr) hit enter then enter in 6 then hit enter.

\( 10P_6 = 151200 \)

a) \( 25P_0 \)
Section 12.9 - Combinations

A combination is a distinct group (or set) of objects without regard to their arrangement.

The number of combinations of $n$ things taken $r$ at a time (that is the number of size $r$ subsets, given a set of size $n$) is written $n\binom{r}$. Since there are $n$ things available and we are choosing $r$ of them, we can read $n\binom{r}$ as “n choose r”. The formula for evaluating numbers of combinations

$$n\binom{r} = \frac{n!}{r!(n-r)!}$$

Permutations are to evaluate the number of arrangements of $n$ things taken $r$ at a time, where repetitions are not allowed, and the order of the items is important.

Combinations are the number of combinations of $n$ things taken $r$ at a time (that is the number of size $r$ subsets, given a set of size $n$), where repetitions are not allowed, and the order is not important.

Factorial Formula for Combinations

The number of combinations, or subsets, of $n$, distinct things taken $r$ at a time, where $r \leq n$, can be calculated as

$$n\binom{r} = \frac{n!}{r!(n-r)!}$$

Alternative Notations are $C(n,r)$ and $\binom{n}{r}$ and $\binom{n}{r}$

Example - A student must select 4 of 7 essay questions to be answered on a test. In how many ways can this selection be made?


**Example** - Toastline Bakery is testing 5 new wheat breads, 4 bran breads and 3 oat breads. If it plans to market 2 of the wheat breads, 2 of the bran breads and one of the oat breads, how many different combinations are possible?

Evaluate each combination

Using a graphing calculator we can perform this calculation directly as follows for a TI-83:

For \(14 \text{C}_6\) enter in 14 then hit MATH – Scroll over to PRB and scroll down to 2 (nCr) hit enter then enter in 6 then hit enter.

\(14 \text{C}_6 = 3003\)

a) \(21 \text{C}_15\)
Section 12.10 - Solving Probability Problems by Using Combinations

**Example** - A club consists of 5 men and 6 women. Four members are to be selected at random to form a committee. What is the probability that the committee will consist of two women?

**Example** - The Honey Bear is testing 10 new flavors of ice cream. They are testing 5 vanilla based, 3 chocolate based and 2 strawberry based ice creams. If we assume that each of the 10 flavors has the same chance of being selected and that 4 new flavors will be produced, find the probability that
a) no chocolate flavors are selected.
b) at least 1 chocolate is selected.
c) 2 vanilla and 2 chocolate are selected.