Chapter 7

Section 7.1 - Introduction to Hypothesis Testing

Objectives:
• State a null hypothesis and an alternative hypothesis
• Identify type I and type II errors and interpret the level of significance
• Determine whether to use a one-tailed or two-tailed statistical test and find a $p$-value
• Make and interpret a decision based on the results of a statistical test
• Write a claim for a hypothesis test

Hypothesis test
• A process that uses sample statistics to test a claim about the value of a population parameter.
• For example: An automobile manufacturer advertises that its new hybrid car has a mean mileage of 50 miles per gallon. To test this claim, a sample would be taken. If the sample mean differs enough from the advertised mean, you can decide the advertisement is wrong.

Statistical hypothesis
• A statement, or claim, about a population parameter.
• Need a pair of hypotheses
  • one that represents the claim
  • the other, its complement
• When one of these hypotheses is false, the other must be true.

Stating a Hypothesis

**Null hypothesis**
• A statistical hypothesis that contains a statement of equality such as $\leq$, $=$, or $\geq$.
• Denoted $H_0$ read “H subzero” or “H naught.”

**Alternative hypothesis**
• A statement of inequality such as $>$, $\neq$, or $<$.
• Must be true if $H_0$ is false.
• Denoted $H_a$ read “H sub-a.”

complementary statements
Stating a Hypothesis

- To write the null and alternative hypotheses, translate the claim made about the population parameter from a verbal statement to a mathematical statement.
- Then write its complement.

\[ H_0 : \mu \leq k \quad H_0 : \mu \geq k \quad H_0 : \mu = k \]
\[ H_a : \mu > k \quad H_a : \mu < k \quad H_a : \mu \neq k \]

- Regardless of which pair of hypotheses you use, you always assume \( \mu = k \) and examine the sampling distribution on the basis of this assumption.

Example: Stating the Null and Alternative Hypotheses
Write the claim as a mathematical sentence. State the null and alternative hypotheses and identify which represents the claim.

1. A school publicizes that the proportion of its students who are involved in at least one extracurricular activity is 61%.
   **Solution:**

2. A car dealership announces that the mean time for an oil change is less than 15 minutes.
   **Solution:**

3. A company advertises that the mean life of its furnaces is more than 18 years
   **Solution:**
Types of Errors

- No matter which hypothesis represents the claim, always begin the hypothesis test assuming that the equality condition in the null hypothesis is true.
- At the end of the test, one of two decisions will be made:
  - reject the null hypothesis
  - fail to reject the null hypothesis
- Because your decision is based on a sample, there is the possibility of making the wrong decision.

<table>
<thead>
<tr>
<th>Decision</th>
<th>$H_0$ is true</th>
<th>$H_0$ is false</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do not reject $H_0$</td>
<td>Correct Decision</td>
<td>Type II Error</td>
</tr>
<tr>
<td>Reject $H_0$</td>
<td>Type I Error</td>
<td>Correct Decision</td>
</tr>
</tbody>
</table>

- A **type I error** occurs if the null hypothesis is rejected when it is true.
- A **type II error** occurs if the null hypothesis is not rejected when it is false.

Example: Identifying Type I and Type II Errors

The USDA limit for salmonella contamination for chicken is 20%. A meat inspector reports that the chicken produced by a company exceeds the USDA limit. You perform a hypothesis test to determine whether the meat inspector’s claim is true. When will a type I or type II error occur? Which is more serious? (Source: United States Department of Agriculture)

Solution:
Level of significance

• Your maximum allowable probability of making a type I error.
  ▪ Denoted by \( \alpha \), the lowercase Greek letter alpha.
• By setting the level of significance at a small value, you are saying that you want the probability of rejecting a true null hypothesis to be small.
• Commonly used levels of significance:
  ▪ \( \alpha = 0.10 \)
  ▪ \( \alpha = 0.05 \)
  ▪ \( \alpha = 0.01 \)
• \( P \) (type II error) = \( \beta \) (beta)

Statistical Tests

• After stating the null and alternative hypotheses and specifying the level of significance, a random sample is taken from the population and sample statistics are calculated.
• The statistic that is compared with the parameter in the null hypothesis is called the test statistic.

<table>
<thead>
<tr>
<th>Population parameter</th>
<th>Test statistic</th>
<th>Standardized test statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>( \bar{x} )</td>
<td>( z ) (Section 7.2 ( n \geq 30 ))</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>( s^2 )</td>
<td>( \chi^2 ) (Section 7.5)</td>
</tr>
<tr>
<td>( p )</td>
<td>( \hat{p} )</td>
<td>( z ) (Section 7.4)</td>
</tr>
</tbody>
</table>

\( P \)-value (or probability value)

• The probability, if the null hypothesis is true, of obtaining a sample statistic with a value as extreme or more extreme than the one determined from the sample data.
• Depends on the nature of the test.

Nature of the Test

• Three types of hypothesis tests
  ▪ left-tailed test
  ▪ right-tailed test
  ▪ two-tailed test
• The type of test depends on the region of the sampling distribution that favors a rejection of \( H_0 \).
• This region is indicated by the alternative hypothesis.
Left-tailed Test
- The alternative hypothesis $H_a$ contains the less-than inequality symbol ($<$).

$H_0$: $\mu \geq k$
$H_a$: $\mu < k$

$P$ is the area to the left of the test statistic.

Right-tailed Test
- The alternative hypothesis $H_a$ contains the greater-than inequality symbol ($>$).

$H_0$: $\mu \leq k$
$H_a$: $\mu > k$

$P$ is the area to the right of the test statistic.

Two-tailed Test
- The alternative hypothesis $H_a$ contains the not equal inequality symbol ($\neq$). Each tail has an area of $\frac{1}{2}P$.

$H_0$: $\mu = k$
$H_a$: $\mu \neq k$

$P$ is twice the area to the left of the negative test statistic.

$P$ is twice the area to the right of the positive test statistic.
Example: Identifying The Nature of a Test

For each claim, state $H_0$ and $H_a$. Then determine whether the hypothesis test is a left-tailed, right-tailed, or two-tailed test. Sketch a normal sampling distribution and shade the area for the $P$-value.

1. A school publicizes that the proportion of its students who are involved in at least one extracurricular activity is 61%.
   Solution:

2. A car dealership announces that the mean time for an oil change is less than 15 minutes.
   Solution:

3. A company advertises that the mean life of its furnaces is more than 18 years.
   Solution:

Making a Decision

Decision Rule Based on $P$-value

- Compare the $P$-value with $\alpha$.
  - If $P \leq \alpha$, then reject $H_0$.
  - If $P > \alpha$, then fail to reject $H_0$.

<table>
<thead>
<tr>
<th>Decision</th>
<th>Claim is $H_0$</th>
<th>Claim is $H_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reject $H_0$</td>
<td>There is enough evidence to reject the claim</td>
<td>There is enough evidence to support the claim</td>
</tr>
<tr>
<td>Fail to reject $H_0$</td>
<td>There is not enough evidence to reject the claim</td>
<td>There is not enough evidence to support the claim</td>
</tr>
</tbody>
</table>

Example: Interpreting a Decision
You perform a hypothesis test for the following claim. How should you interpret your decision if you reject $H_0$? If you fail to reject $H_0$?

1. $H_0$ (Claim): A school publicizes that the proportion of its students who are involved in at least one extracurricular activity is 61%.
   - The claim is represented by $H_0$.
   - If you reject $H_0$ you should conclude “there is sufficient evidence to indicate that the school’s claim is false.”
   - If you fail to reject $H_0$ (or Retain $H_0$), you should conclude “there is insufficient evidence to indicate that the school’s claim (proportion of students who are involved in at least one extracurricular activity) is false.”

2. $H_a$ (Claim): A car dealership announces that the mean time for an oil change is less than 15 minutes.
   - The claim is represented by $H_a$.
   - $H_0$ is “the mean time for an oil change is greater than or equal to 15 minutes.”
   - If you reject $H_0$ you should conclude “there is enough evidence to support the dealership’s claim that the mean time for an oil change is less than 15 minutes.”
   - If you fail to reject $H_0$ (or Retain $H_0$), you should conclude “there is not enough evidence to support dealership’s claim that the mean time for an oil change is less than 15 minutes.”

Steps for Hypothesis Testing
1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.
   $H_0$: ?  $H_a$: ?
2. Specify the level of significance.
   $\alpha$ = ?
3. Determine the standardized sampling distribution and draw its graph.
4. Calculate the test statistic and its standardized value.
   Add it to your sketch.
5. Find the $P$-value.
6. Use the following decision rule.
   
   Is the $P$-value less than or equal to the level of significance?
   
   Yes → Reject $H_0$.
   No → Fail to reject $H_0$.

7. Write a statement to interpret the decision in the context of the original claim.
Section 7.2 - Hypothesis Testing for the Mean (Large Samples)

Objectives:
1. Find P-values and use them to test a mean $\mu$
2. Use P-values for a z-test
3. Find critical values and rejection regions in a normal distribution
4. Use rejection regions for a z-test

Using P-values to Make a Decision

Decision Rule Based on P-value
- To use a P-value to make a conclusion in a hypothesis test, compare the P-value with $\alpha$.
  1. If $P \leq \alpha$, then reject $H_0$.
  2. If $P > \alpha$, then fail to reject $H_0$ (or Retain $H_0$)

Example: Interpreting a P-value
The P-value for a hypothesis test is $P = 0.0237$. What is your decision if the level of significance is
1. 0.05?
   Solution:
2. 0.01?
   Solution:

Finding the P-value
After determining the hypothesis test’s standardized test statistic and the test statistic’s corresponding area, do one of the following to find the P-value.
- For a left-tailed test, $P =$ (Area in left tail).
- For a right-tailed test, $P =$ (Area in right tail).
- For a two-tailed test, $P = 2(Area in tail of test statistic)$. 
Example: Finding the P-value

1. Find the P-value for a left-tailed hypothesis test with a test statistic of $z = -2.23$. Decide whether to reject $H_0$ if the level of significance is $\alpha = 0.01$.
   
   Solution:

2. Find the P-value for a two-tailed hypothesis test with a test statistic of $z = 2.14$. Decide whether to reject $H_0$ if the level of significance is $\alpha = 0.05$.
   
   Solution:

Z-Test for a Mean $\mu$

- Can be used when the population is normal and $\sigma$ is known, or for any population when the sample size $n$ is at least 30.
- The test statistic is the sample mean
- The standardized test statistic is $z$

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

- When $n \geq 30$, the sample standard deviation $s$ can be substituted for $\sigma$. 

Using \( P \)-values for a \( z \)-Test for Mean \( \mu \)

<table>
<thead>
<tr>
<th><strong>In Words</strong></th>
<th><strong>In Symbols</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.</td>
<td>State ( H_0 ) and ( H_a ).</td>
</tr>
<tr>
<td>2. Specify the level of significance.</td>
<td>Identify ( \alpha ).</td>
</tr>
<tr>
<td>3. Determine the standardized test statistic.</td>
<td>( z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} )</td>
</tr>
<tr>
<td>4. Find the area that corresponds to ( z ).</td>
<td>Use Table 4 in Appendix B.</td>
</tr>
<tr>
<td>5. Find the ( P )-value. a. For a left-tailed test, ( P = (\text{Area in left tail}) ). b. For a right-tailed test, ( P = (\text{Area in right tail}) ). c. For a two-tailed test, ( P = 2(\text{Area in tail of test statistic}) ).</td>
<td>Reject ( H_0 ) if ( P )-value is less than or equal to ( \alpha ). Otherwise, fail to reject ( H_0 ).</td>
</tr>
<tr>
<td>6. Make a decision to reject or fail to reject the null hypothesis.</td>
<td></td>
</tr>
<tr>
<td>7. Interpret the decision in the context of the original claim.</td>
<td></td>
</tr>
</tbody>
</table>

**Example: Hypothesis Testing Using \( P \)-values**

In auto racing, a pit crew claims that its mean pit stop time (for 4 new tires and fuel) is less than 13 seconds. A random selection of 32 pit stop times has a sample mean of 12.9 seconds and a standard deviation of 0.19 seconds. Is there enough evidence to support the claim at \( \alpha = 0.01 \)? Use a \( P \)-value.

**Solution:**
Example: Hypothesis Testing Using $P$-values

The National Institute of Diabetes and Digestive and Kidney Diseases reports that the average cost of bariatric (weight loss) surgery is $22,500. You think this information is incorrect. You randomly select 30 bariatric surgery patients and find that the average cost for their surgeries is $21,545 with a standard deviation of $3015. Is there enough evidence to support your claim at $\alpha = 0.05$? Use a $P$-value.

Solution:

Rejection Regions and Critical Values

Rejection region (or critical region)

- The range of values for which the null hypothesis is not probable.
- If a test statistic falls in this region, the null hypothesis is rejected.
- A critical value $z_0$ separates the rejection region from the nonrejection region.

Finding Critical Values in a Normal Distribution

1. Specify the level of significance $\alpha$.
2. Decide whether the test is left-, right-, or two-tailed.
3. Find the critical value(s) $z_0$. If the hypothesis test is
   a. *left-tailed*, find the $z$-score that corresponds to an area of $\alpha$,
   b. *right-tailed*, find the $z$-score that corresponds to an area of $1 - \alpha$,
   c. *two-tailed*, find the $z$-score that corresponds to $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$.
4. Sketch the standard normal distribution. Draw a vertical line at each critical value and shade the rejection region(s).
Example: Finding Critical Values

Find the critical value and rejection region for a two-tailed test with $\alpha = 0.05$.

Solution:

**Decision Rule Based on Rejection Region**

To use a rejection region to conduct a hypothesis test, calculate the standardized test statistic, $z$. If the standardized test statistic

1. is in the rejection region, then reject $H_0$.
2. is *not* in the rejection region, then fail to reject $H_0$.

<table>
<thead>
<tr>
<th>Level of Confidence $c$</th>
<th>Critical Value $z_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70, or 70%</td>
<td>1.04</td>
</tr>
<tr>
<td>0.75, or 75%</td>
<td>1.15</td>
</tr>
<tr>
<td>0.80, or 80%</td>
<td>1.28</td>
</tr>
<tr>
<td>0.85, or 85%</td>
<td>1.44</td>
</tr>
<tr>
<td>0.90, or 90%</td>
<td>1.645</td>
</tr>
<tr>
<td>0.95, or 95%</td>
<td>1.96</td>
</tr>
<tr>
<td>0.98, or 98%</td>
<td>2.33</td>
</tr>
<tr>
<td>0.99, or 99%</td>
<td>2.58</td>
</tr>
</tbody>
</table>
Using Rejection Regions for a z-Test for a Mean \( \mu \)

<table>
<thead>
<tr>
<th>In Words</th>
<th>In Symbols</th>
</tr>
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<tbody>
<tr>
<td>1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.</td>
<td>State ( H_0 ) and ( H_a ).</td>
</tr>
<tr>
<td>2. Specify the level of significance.</td>
<td>Identify ( \alpha ).</td>
</tr>
<tr>
<td>3. Sketch the sampling distribution.</td>
<td></td>
</tr>
<tr>
<td>4. Determine the critical value(s).</td>
<td>Use Table 4 in Appendix B.</td>
</tr>
<tr>
<td>5. Determine the rejection region(s).</td>
<td></td>
</tr>
</tbody>
</table>
| 6. Find the standardized test statistic.                               | \[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \text{ or if } n \geq 30 \]  
                                 | \[ \text{use } \sigma \approx s. \]                                    |
| 7. Make a decision to reject or fail to reject the null hypothesis.     | If \( z \) is in the rejection region, reject \( H_0 \).                 
                                 | Otherwise, fail to reject \( H_0 \).                                    |
| 8. Interpret the decision in the context of the original claim.         |                                                                           |

**Example: Testing with Rejection Regions**

Employees at a construction and mining company claim that the mean salary of the company’s mechanical engineers is less than that of one of its competitors, which is $68,000. A random sample of 30 of the company’s mechanical engineers has a mean salary of $66,900 with a standard deviation of $5500. At \( \alpha = 0.05 \), test the employees’ claim.

**Solution:**
Example: Testing with Rejection Regions
The U.S. Department of Agriculture claims that the mean cost of raising a child from birth to age 2 by husband-wife families in the U.S. is $13,120. A random sample of 500 children (age 2) has a mean cost of $12,925 with a standard deviation of $1745. At $\alpha = 0.10$, is there enough evidence to reject the claim?
(Adapted from U.S. Department of Agriculture Center for Nutrition Policy and Promotion)
Solution:
Section 7.3 - Hypothesis Testing for the Mean (Small Samples)

Objectives:
- Find critical values in a $t$-distribution
- Use the $t$-test to test a mean $\mu$
- Use technology to find $P$-values and use them with a $t$-test to test a mean $\mu$

Finding Critical Values in a $t$-Distribution
1. Identify the level of significance $\alpha$.
2. Identify the degrees of freedom d.f. = $n - 1$.
3. Find the critical value(s) using Table 5 in Appendix B in the row with $n - 1$ degrees of freedom. If the hypothesis test is
   a. left-tailed, use “One Tail, $\alpha$” column with a negative sign,
   b. right-tailed, use “One Tail, $\alpha$” column with a positive sign,
   c. two-tailed, use “Two Tails, $\alpha$” column with a negative and a positive sign.

Example: Finding Critical Values for $t$
1. Find the critical value $t_0$ for a left-tailed test given $\alpha = 0.05$ and $n = 21$.
   Solution:

2. Find the critical values $-t_0$ and $t_0$ for a two-tailed test given $\alpha = 0.10$ and $n = 26$.
   Solution:
t-Test for a Mean $\mu$ ($n < 30$, $\sigma$ Unknown)

**t-Test for a Mean**

- A statistical test for a population mean.
- The $t$-test can be used when the population is normal or nearly normal, $\sigma$ is unknown, and $n < 30$.
- The **test statistic** is the sample mean
- The **standardized test statistic** is $t$.

\[
t = \frac{\bar{x} - \mu}{s/\sqrt{n}}
\]

- The degrees of freedom are $d.f. = n - 1$.

**Using the t-Test for a Mean $\mu$ (Small Sample)**

<table>
<thead>
<tr>
<th><strong>In Words</strong></th>
<th><strong>In Symbols</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.</td>
<td>State $H_0$ and $H_a$.</td>
</tr>
<tr>
<td>2. Specify the level of significance.</td>
<td>Identify $\alpha$.</td>
</tr>
<tr>
<td>3. Identify the degrees of freedom and sketch the sampling distribution.</td>
<td>$d.f. = n - 1$.</td>
</tr>
<tr>
<td>4. Determine any critical value(s).</td>
<td>Use Table 5 in Appendix B.</td>
</tr>
<tr>
<td>5. Determine any rejection region(s).</td>
<td></td>
</tr>
<tr>
<td>6. Find the standardized test statistic.</td>
<td>$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$</td>
</tr>
<tr>
<td>7. Make a decision to reject or fail to reject the null hypothesis.</td>
<td>If $t$ is in the rejection region, reject $H_0$. Otherwise, fail to reject $H_0$.</td>
</tr>
<tr>
<td>8. Interpret the decision in the context of the original claim.</td>
<td></td>
</tr>
</tbody>
</table>
Example: Testing $\mu$ with a Small Sample

A used car dealer says that the mean price of a 2008 Honda CR-V is at least $20,500. You suspect this claim is incorrect and find that a random sample of 14 similar vehicles has a mean price of $19,850 and a standard deviation of $1084. Is there enough evidence to reject the dealer’s claim at $\alpha = 0.05$? Assume the population is normally distributed. (*Adapted from Kelley Blue Book*)

Example: Testing $\mu$ with a Small Sample

An industrial company claims that the mean pH level of the water in a nearby river is 6.8. You randomly select 19 water samples and measure the pH of each. The sample mean and standard deviation are 6.7 and 0.24, respectively. Is there enough evidence to reject the company’s claim at $\alpha = 0.05$? Assume the population is normally distributed.

Solution: