Chapter 2 – Functions and Graphs

Section 2.1 – Functions

Objectives:

- The student will be able to do point-by-point plotting of equations in two variables.
- The student will be able to give and apply the definition of a function.
- The student will be able to identify domain and range of a function.
- The student will be able to use function notation.
- The student will be able to solve applications

Graphing an Equation

- To sketch the graph an equation in $x$ and $y$, we need to find ordered pairs that solve the equation and plot the ordered pairs on a grid. This process is called **point-by-point plotting**.

For example, let’s plot the graph of the equation $y = x^2 + 2$. This is the graph of $y = x^2$ shifted vertically up 2 units.
Review Transformations:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Equation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>$f(x) = b^x + c$</td>
<td>Shifted up $c$ units</td>
</tr>
<tr>
<td></td>
<td>$f(x) = b^x - c$</td>
<td>Shitted down $c$ units</td>
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<tr>
<td>Horizontal</td>
<td>$f(x) = b^{x+c}$</td>
<td>Shifted left $c$ units</td>
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<td></td>
<td>$f(x) = b^{x-c}$</td>
<td>Shifted right $c$ units</td>
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<tr>
<td>Reflection</td>
<td>$f(x) = -b^x$</td>
<td>Reflect across x-axis</td>
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<td>$f(x) = b^{-x}$</td>
<td>Reflect across y-axis</td>
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<tr>
<td>Vertical Stretching or Shrinking</td>
<td>$f(x) = cb^x$</td>
<td>Stretch if $c&gt;1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shrink if $0&lt;c&lt;1$</td>
</tr>
<tr>
<td>Horizontal Stretching or Shrinking</td>
<td>$f(x) = b^{cx}$</td>
<td>Stretch if $0&lt;c&lt;1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shrink if $c&gt;1$</td>
</tr>
</tbody>
</table>

A relation is a set of ordered pairs of real numbers.

The domain, $D$, of a relation is the set of all first coordinates of the ordered pairs in the relation (the xs).

The range, $R$, of a relation is the set of all second coordinates of the ordered pairs in the relation (the ys).

In graphing relations, the horizontal axis is called the domain axis and the vertical axis is called the range axis.

The domain and range of a relation can often be determined from the graph of the relation.

A function is a special kind of relation that pairs each element of the domain with one and only one element of the range. (For every $x$ there is exactly one $y$.) A function is a correspondent between a first set, domain, and a second set, range.
In a function no two ordered pairs have the same first coordinate. That is, each first coordinate appears only once. Although every function is by definition a relation, not every relation is a function.

To determine whether or not the graph of a relation represents a function, we apply the **vertical line test** which states that if any vertical line intersects the graph of a relation in more than one point, then the relation graphed is **not** a function.

- **Visualization of Mapping**

  ![Mapping Diagram](image)

  *F* is a function.  

  *G* is not a function.

**Example:** Is the relation a function?
Functions Specified by Equations

- If in an equation in two variables, we get exactly one output (value for the dependent variable) for each input (value for the independent variable), then the equation specifies a function. The graph of such a function is just the graph of the specifying equation.
- If we get more than one output for a given input, the equation does not specify a function.
- If you have the graph of an equation, there is an easy way to determine if it is the graph of a function. It is called the **vertical line test** which states that:
  - An equation specifies a function if each vertical line in the coordinate system passes through at most one point on the graph of the equation.
  - If any vertical line passes through two or more points on the graph of an equation, then the equation does not specify a function.

This graph is not the graph of a function because you can draw a vertical line which crosses it twice.

This is the graph of a function because any vertical line crosses only once.
Function Notation
- The following notation is used to describe functions. The variable y will now be called $f(x)$.
- This is read as “$f$ of $x$” and simply means the $y$ coordinate of the function corresponding to a given $x$ value.
  For example this equation $y = x^2 - 2$ can now be expressed as $f(x) = x^2 - 2$

Function Evaluation
Example: Consider our function $f(x) = x^2 - 2$
What does $f(-3)$ mean? Or said another way what is the value of $y$ when $x=-3$?

Solution: Replace $x$ with the value $-3$ and evaluate the expression
  The result is 11. This means that the point $(-3, 11)$ is on the graph of the function.

Example: Consider the function $f(x) = \sqrt{3x} - 2$
  a) Find $f(a)$

b) Find $f(2)$

c) Find $f(6+h)$
Domain of a Function
The domain of a function is the values of $x$ for which the function is defined

How to find the domain?
Ask yourself two questions:
1. Do I have any variables in my denominator? This question is important because if you have a denominator equal to zero then your function is undefined at that value of the variable.
2. Do I have any even indexed roots? This question is important because if you have an even indexed root you cannot take an even indexed root of a negative value.

Example: Find the domain of the function

$f(x) = \sqrt{\frac{1}{2}x - 4}$
Mathematical Modeling

Example: The price-demand function for a company is given by

\[ p(x) = 1000 - 5x, \quad 0 \leq x \leq 100 \]

where \( x \) represents the number of items and \( P(x) \) represents the price of the item. Determine the revenue function and find the revenue generated if 50 items are sold.

Solution: Revenue = Price \cdot Quantity, so

\[ R(x) = p(x) \cdot x = (1000 - 5x) \cdot x \]

When 50 items are sold, \( x = 50 \), so we will evaluate the revenue function at \( x = 50 \):

\[ R(50) = \left(1000 - 5(50)\right) \cdot 50 = 37,500 \]

The domain of the function has already been specified. We are told that \( 0 \leq x \leq 100 \).
Break-Even and Profit-Loss Analysis

- Any manufacturing company has **costs** $C$ and **revenues** $R$.
- The company will have a **loss** if $R < C$, will **break even** if $R = C$, and will have a **profit** if $R > C$.
- Costs include **fixed costs** such as plant overhead, etc. and **variable costs**, which are dependent on the number of items produced. $C = a + bx$ ($x$ is the number of items produced)
- **Price-demand** functions, usually determined by financial departments, play an important role in profit-loss analysis.
  \[ p = m - nx \]
  ($x$ is the number of items than can be sold at $p$ per item.)
- The **revenue function** is
  \[ R = (\text{number of items sold}) \cdot (\text{price per item}) = xp = x(m - nx) \]
- The **profit function** is
  \[ P = R - C = x(m - nx) - (a + bx) \]

Example of Profit-Loss Analysis

A company manufactures notebook computers. Its marketing research department has determined that the data is modeled by the price-demand function $p(x) = 2,000 - 60x$, when $1 \leq x \leq 25$, ($x$ is in thousands).

What is the company’s revenue function and what is its domain?

**Solution:** Since Revenue = Price $\cdot$ Quantity,
\[ R(x) = x \cdot p(x) = x \cdot (2000 - 60x) = 2000x - 60x^2 \]

The domain of this function is the same as the domain of the price-demand function, which is $1 \leq x \leq 25$ (in thousands.)
**Profit Problem**

The financial department for the company in the preceding problem has established the following cost function for producing and selling $x$ thousand notebook computers:

$$C(x) = 4000 + 500x \quad (x \text{ is in thousand dollars}).$$

Write a profit function for producing and selling $x$ thousand notebook computers, and indicate the domain of this function.

**Solution:**

Since Profit = Revenue − Cost, and our revenue function from the preceding problem was

$$R(x) = 2000x - 60x^2,$$

$$P(x) = R(x) - C(x) = 2000x - 60x^2 - (4000 + 500x)$$

$$= -60x^2 + 1500x - 4000.$$

The domain of this function is the same as the domain of the original price-demand function, $1 \leq x \leq 25$ (in thousands.)
Section 2.2 - Elementary Functions: Graphs and Transformations

Objectives:
- The student will become familiar with a beginning library of elementary functions.
- The student will be able to transform functions using vertical and horizontal shifts.
- The student will be able to transform functions using reflections, stretches, and shrinks.
- The student will be able to graph piecewise-defined functions.

**Identity Function**
\[ f(x) = x \]

**Square Function**
\[ h(x) = x^2 \]

**Cube Function**
\[ m(x) = x^3 \]

**Square Root Function**
\[ n(x) = \sqrt{x} \]

**Cube Root Function**
\[ p(x) = \sqrt[3]{x} \]

**Absolute Value Function**
\[ p(x) = |x| \]
Vertical Shift
The graph of $y = f(x) + k$ can be obtained from the graph of $y = f(x)$ by **vertically translating** (shifting) the graph of the latter upward $k$ units if $k$ is positive and downward $|k|$ units if $k$ is negative.

**Example:** Graph $y = |x|$, $y = |x| + 4$, and $y = |x| - 5$.

![Graph of vertical shifts](image)

Horizontal Shift
The graph of $y = f(x + h)$ can be obtained from the graph of $y = f(x)$ by **horizontally translating** (shifting) the graph of the latter $h$ units to the left if $h$ is positive and $|h|$ units to the right if $h$ is negative.

**Example:** Graph $y = |x|$, $y = |x + 4|$, and $y = |x - 5|$.

![Graph of horizontal shifts](image)
Reflection, Stretches and Shrinks

- The graph of \( y = Af(x) \) can be obtained from the graph of \( y = f(x) \) by multiplying each ordinate value by \( A \). (The ordinate or \( y \) coordinate)
- If \( A > 1 \), the result is a **vertical stretch** of the graph of \( y = f(x) \).
- If \( 0 < A < 1 \), the result is a **vertical shrink** of the graph of \( y = f(x) \).
- If \( A = -1 \), the result is a **reflection in the x axis**.
- Graph \( y = |x|, \ y = 2|x|, \ y = 0.5|x|, \) and \( y = -2|x| \).

![Graphs of |x|, 2|x|, 0.5|x|, -2|x|](image)

**Summary of Graph Transformations**

- **Vertical Translation**: \( y = f(x) + k \)
  - \( k > 0 \) Shift graph of \( y = f(x) \) up \( k \) units.
  - \( k < 0 \) Shift graph of \( y = f(x) \) down \( |k| \) units.
- **Horizontal Translation**: \( y = f(x + h) \)
  - \( h > 0 \) Shift graph of \( y = f(x) \) left \( h \) units.
  - \( h < 0 \) Shift graph of \( y = f(x) \) right \( |h| \) units.
- **Reflection**: \( y = -f(x) \) Reflect the graph of \( y = f(x) \) in the \( x \) axis.
- **Vertical Stretch and Shrink**: \( y = Af(x) \)
  - \( A > 1 \): Stretch graph of \( y = f(x) \) vertically by multiplying each ordinate value by \( A \).
  - \( 0 < A < 1 \): Shrink graph of \( y = f(x) \) vertically by multiplying each ordinate value by \( A \).
### Piecewise-Defined Functions

- Earlier we noted that the absolute value of a real number $x$ can be defined as

$$|x| = \begin{cases} 
x & \text{if } x < 0 \\
-x & \text{if } x > 0 
\end{cases}$$

- Notice that this function is defined by different rules for different parts of its domain. Functions whose definitions involve more than one rule are called **piecewise-defined** functions.
- Graphing one of these functions involves graphing each rule over the appropriate portion of the domain.

**Example:** Graph the following

$$f(x) = \begin{cases} 
2 - 2x & \text{if } x < 2 \\
x - 2 & \text{if } x \geq 2
\end{cases}$$

**Solution:**

![Graph of the piecewise function](image)

Notice that the point (2,0) is included but the point (2, -2) is not.
Section 2.3 - Quadratic Functions

Objectives:
- The student will be able to identify and define quadratic functions, equations, and inequalities.
- The student will be able to identify and use properties of quadratic functions and their graphs.
- The student will be able to solve applications of quadratic functions.

Quadratic Functions
If $a, b, c$ are real numbers with $a$ not equal to zero, then the function $f(x) = ax^2 + bx + c$ is a quadratic function and its graph is a parabola.

Vertex Form of the Quadratic Function
It is convenient to convert the general form of a quadratic equation $f(x) = ax^2 + bx + c$ to what is known as the vertex form: $f(x) = a(x - h)^2 + k$
Completing the Square to Find the Vertex of a Quadratic Function
Consider \( f(x) = -3x^2 + 6x - 1 \)
Complete the square to find the vertex.

Solution:
Factor the coefficient of \( x^2 \) out of the first two terms:
\[
f(x) = -3(x^2 - 2x) - 1
\]
Add \( \left( \frac{2}{2} \right)^2 = \left( -1 \right)^2 = 1 \). Add 1 to complete the square inside the parentheses. Because of the \(-3\) outside the parentheses, we have actually added \(-3\), so we must add +3 to the outside.
\[
f(x) = -3(x^2 - 2x + 1) - 1 + 3
\]
\[
f(x) = -3(x - 1)^2 + 2
\]
The vertex is \((1, 2)\)
The quadratic function opens down since the coefficient of the \(x^2\) term is \(-3\), which is negative.

Intercepts of a Quadratic Function
Example: Find the \( x \) and \( y \) intercepts of \( f(x) = -3x^2 + 6x - 1 \)
Solution:
\[ x \text{ intercepts: } \text{Set } f(x) = 0: 0 = -3x^2 + 6x - 1 \]
Use the quadratic formula: \[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(-3)(-1)}}{2(-3)} = \frac{-6 \pm \sqrt{24}}{-6} \approx 0.184 \text{ and } 1.816
\]
\[ y \text{ intercept: } \text{Let } x = 0. \text{ If } x = 0, \text{ then } y = -1, \text{ so } (0, -1) \text{ is the } y \text{ intercept.}
\]
\[ f(0) = -3(0)^2 + 6(0) - 1 = -1 \]
Generalization
For \( f(x) = a(x - h)^2 + k \)
- If \( a \neq 0 \), then the graph of \( f \) is a parabola.
  - If \( a > 0 \), the graph opens upward.
  - If \( a < 0 \), the graph opens downward. Vertex is \( (h, k) \)
- Axis of symmetry: \( x = h \)
- \( f(h) = k \) is the minimum if \( a > 0 \), otherwise the maximum
- Domain = set of all real numbers
- Range: \{ \( y \mid y \leq k \} \) if \( a < 0 \). If \( a > 0 \), the range is \{ \( y \mid y \geq k \} \)

Solving Quadratic Inequalities
**Example:** Solve the quadratic inequality \( -x^2 + 5x + 3 > 0 \)

**Solution:** This inequality holds for those values of \( x \) for which the graph of \( f(x) \) is at or above the \( x \) axis. This happens for \( x \) between the two \( x \) intercepts, including the intercepts. Thus, the solution set for the quadratic inequality is \( -0.5414 < x < 5.5414 \) or \([-0.5414, 5.5414] \)
Application of Quadratic Functions

A Macon, Georgia, peach orchard farmer now has 20 trees per acre. Each tree produces, on the average, 300 peaches. For each additional tree that the farmer plants, the number of peaches per tree is reduced by 10. How many more trees should the farmer plant to achieve the maximum yield of peaches? What is the maximum yield?

Solution:

Yield = (number of peaches per tree) × (number of trees)

Yield = 300 × 20 = 6000 (currently)

Plant one more tree: Yield = (300 – 1(10)) × (20 + 1) = 290 × 21 = 6090 peaches.

Plant two more trees:

Yield = (300 – 2(10)) × (20 + 2) = 280 × 22 = 6160

Let x represent the number of additional trees.

Then Yield = (300 – 10x) (20 + x) = −10x² + 100x + 6000

To find the maximum yield, note that the Y(x) function is a quadratic function opening downward. Hence, the vertex of the function will be the maximum value of the yield.

Complete the square to find the vertex of the parabola:

Add \( \left( \frac{b}{2} \right)^2 = \left( \frac{-10}{2} \right)^2 = (-5)^2 = 25 \). Add 25 to complete the square inside the parentheses

\[ Y(x) = -10(x^2 - 10x + 25) + 6000 + 250 \]

We have to add 250 on the outside since we multiplied –10 by 25 = −250.

\[ Y(x) = -10(x - 5)^2 + 6250 \]

Thus, the vertex of the quadratic function is (5, 6250). So, the farmer should plant 5 additional trees and obtain a yield of 6250 peaches. We know this yield is the maximum of the quadratic function since the value of a is −10. The function opens downward, so the vertex must be the maximum.
### Break-Even Analysis

The financial department of a company that produces digital cameras has the revenue and cost functions for \( x \) million cameras are as follows:

\[
R(x) = x(94.8 - 5x)
\]

\[
C(x) = 156 + 19.7x.
\]

Both have domain \( 1 \leq x \leq 15 \)

Break-even points are the production levels at which \( R(x) = C(x) \). Find the break-even points algebraically to the nearest thousand cameras.

#### Solution:

Set \( R(x) \) equal to \( C(x) \):

\[
x(94.8 - 5x) = 156 + 19.7x
\]

\[
-5x^2 + 75.1x - 156 = 0
\]

\[
x = \frac{-75.1 \pm \sqrt{75.1^2 - 4(-6)(-156)}}{2(-5)}
\]

\[
x = 2.490 \text{ or } 12.530
\]

The company breaks even at \( x = 2.490 \) and 12.530 million cameras.

If we graph the cost and revenue functions on a graphing utility, we obtain the following graphs, showing the two intersection points:
Section 2.4 - Polynomial and Rational Functions

Objectives:
- The student will be able to graph and identify properties of polynomial functions.
- The student will be able to calculate polynomial regression using a calculator.
- The student will be able to graph and identify properties of rational functions.
- The student will be able to solve applications of polynomial and rational functions.

Polynomial Functions
A polynomial function is a function that can be written in the form
\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]
for \( n \) a nonnegative integer, called the degree of the polynomial. The domain of a polynomial function is the set of all real numbers. A polynomial of degree 0 is a constant. A polynomial of degree 1 is a linear function. A polynomial of degree 2 is a quadratic function.

Shapes of Polynomials
- A polynomial is called odd if it only contains odd powers of \( x \)
- It is called even if it only contains even powers of \( x \)
Observations Odd Polynomials

- For an odd polynomial,
  - the graph is symmetric about the origin
  - the graphs starts negative, ends positive, or vice versa, depending on whether the leading coefficient is positive or negative
  - either way, a polynomial of degree $n$ crosses the $x$ axis at least once, at most $n$ times.

Observations Even Polynomials

- For an even polynomial,
  - the graph is symmetric about the $y$ axis
  - the graphs starts negative, ends negative, or starts and ends positive, depending on whether the leading coefficient is positive or negative
  - either way, a polynomial of degree $n$ crosses the $x$ axis at most $n$ times. It may or may not cross at all.

Let’s look at the shapes of some even and odd polynomials
Look for some of the following properties:
- Symmetry
- Number of $x$ axis intercepts
- Number of local maxima/minima
- What happens as $x$ goes to $+\infty$ or $-\infty$?
$f(x) = x - 2$

$n=1$  ODD
Symmetry: Origin
Number of $x$ axis intercepts: 1
Number of local maxima/minima: None
What happens as $x$ goes to $+\infty$? Increases
What happens as $x$ goes to $-\infty$? Decreases

$g(x) = x^3 - 2x$

$n=3$  ODD
Symmetry: Origin
Number of $x$ axis intercepts: 2
Local Maxima: (-1,1)  Local Minima: (1,-1)
What happens as $x$ goes to $+\infty$? Increases w/o bound
What happens as $x$ goes to $-\infty$? Decreases w/o bound

$h(x) = x^5 - 5x^3 + 4x + 1$

$n=5$  ODD
Symmetry: Origin
Number of $x$ axis intercepts: 5
Number of local maxima/minima: 2 of each
What happens as $x$ goes to $+\infty$? Increases w/o bound
What happens as $x$ goes to $-\infty$? Decreases w/o bound

$F(x) = x^2 - 2x + 2$

$n=2$  Even
Symmetry: $y$-axis
Number of $x$ axis intercepts: None
Number of local maxima/minima: None
What happens as $x$ goes to $+\infty$? Increases w/o bound
What happens as $x$ goes to $-\infty$? Increases w/o bound
Characteristics of Polynomials

- Graphs of polynomials are continuous. One can sketch the graph without lifting up the pencil.
- Graphs of polynomials have no sharp corners.
- Graphs of polynomials usually have turning points, which is a point that separates an increasing portion of the graph from a decreasing portion. The number of turning points is at most \(n-1\).
- A polynomial of degree \(n\) can have at most \(n\) linear factors. Therefore, the graph of a polynomial function of positive degree \(n\) can intersect the \(x\) axis at most \(n\) times.
- A polynomial of degree \(n\) may intersect the \(x\) axis fewer than \(n\) times.

\[G(x) = 2x^4 - 4x^2 + x - 1\]

\[H(x) = x^6 - 7x^4 + 14x^2 - x - 5\]

- \(n=4\) Even
  - Symmetry: \(y\)-axis
  - Number of \(x\) axis intercepts: 2
  - Number of local maxima/minima: 1 max; 2 min
  - What happens as \(x\) goes to \(+\infty\)? Increases w/o bound
  - What happens as \(x\) goes to \(-\infty\)? Increases w/o bound

- \(n=6\) Even
  - Symmetry: \(y\)-axis
  - Number of \(x\) axis intercepts: 4
  - Number of local maxima/minima: 2 max; 3 min
  - What happens as \(x\) goes to \(+\infty\)? Increases w/o bound
  - What happens as \(x\) goes to \(-\infty\)? Increases w/o bound
Quadratic Regression

A visual inspection of the plot of a data set might indicate that a parabola would be a better model of the data than a straight line. In that case, rather than using linear regression to fit a linear model to the data, we would use **quadratic regression** on a graphing calculator to find the function of the form \( y = ax^2 + bx + c \) that best fits the data.

Example of Quadratic Regression

An automobile tire manufacturer collected the data in the table relating tire pressure \( x \) (in pounds per square inch) and mileage (in thousands of miles.) Using quadratic regression on a graphing calculator, find the quadratic function that best fits the data.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Mileage</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>45</td>
</tr>
<tr>
<td>30</td>
<td>52</td>
</tr>
<tr>
<td>32</td>
<td>55</td>
</tr>
<tr>
<td>34</td>
<td>51</td>
</tr>
<tr>
<td>36</td>
<td>47</td>
</tr>
</tbody>
</table>

**Solution:**

Enter the data in a graphing calculator and obtain the lists below. Choose quadratic regression from the statistics menu and obtain the coefficients as shown:

\[
\begin{align*}
&\text{QuadReg} \\
&y=ax^2+bx+c \\
&a=-0.5178571429 \\
&b=33.29285714 \\
&c=-480.9428571
\end{align*}
\]

This means that the equation that best fits the data is: \( y = -0.517857x^2 + 33.292857x - 480.942857 \)
Rational Functions

- A rational function \( f(x) \) is a quotient of two polynomials, \( n(x) \) and \( d(x) \), for all \( x \) such that \( d(x) \) is not equal to zero.
- Example: Let \( p(x) = x + 5 \) and \( q(x) = x - 2 \), then \( f(x) = \frac{x+5}{x-2} \) is a rational function whose domain is all real values of \( x \) with the exception of 2 (Why?)

At the value of 2 the denominator becomes zero and thus \( f(x) \) is undefined at \( x=2 \).

Vertical Asymptotes of Rational Functions

\( x \) values at which the function is undefined represent vertical asymptotes to the graph of the function. A vertical asymptote is a line of the form \( x = k \) which the graph of the function approaches but does not cross. In the figure below, which is the graph of \( f(x) = \frac{x+5}{x-2} \) the line \( x = 2 \) is a vertical asymptote.

Said another way vertical asymptotes occur where the denominator equals zeros.

Horizontal Asymptotes of Rational Functions

A horizontal asymptote of a function is a line of the form \( y = k \) which the graph of the function approaches as \( x \) approaches \( \pm \infty \).

Determining a Vertical Asymptotes – For a rational function \( f(x) = \frac{p(x)}{q(x)} \), if \( a \) is a zero of the denominator, then the line \( x = a \) is a vertical asymptote.

Determining a Horizontal Asymptotes – For a rational function

\[
f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0}; \quad a_n \neq 0, b_m \neq 0,
\]

the degree of the numerator is \( n \) and the degree of the denominator is \( m \).

1. if \( n < m \), the \( x \)-axis, or \( y = 0 \), is the horizontal asymptote of the graph of \( f \).
2. if \( n = m \), the line, or \( y = \frac{a_n}{b_m} \), is the horizontal asymptote of the graph of \( f \).
3. if \( n > m \), the graph has no horizontal asymptote.
Generalizations about Asymptotes of Rational Functions

**Vertical Asymptotes:**
Case 1: Suppose \( p(x) \) and \( q(x) \) have no real zero in common. The line \( x = c \) is a vertical asymptote if \( q(c) = 0 \).
Case 2: If \( p(x) \) and \( d(x) \) have one or more real zeros in common, cancel the linear factors. Then apply Case 1.

**Horizontal Asymptotes:**
Case 1: If degree of \( p(x) \) < degree of \( q(x) \) then \( y = 0 \) is the horizontal asymptote.
Case 2: If degree of \( p(x) = degree \) of \( q(x) \) then \( y = \frac{a_n}{b_m} \), is the horizontal asymptote, where \( a \) is the leading coefficient of \( p(x) \) and \( b \) is the leading coefficient of \( q(x) \).
Case 3: If degree of \( p(x) > degree \) of \( q(x) \) there is no horizontal asymptote.

**Bounded**
A function \( f \) is **bounded** if the entire graph of \( f \) lies between two horizontal lines.
The only polynomials that are bounded are the constant functions, but there are many rational functions that are bounded.
Application of Rational Functions
A company that manufactures computers has established that, on the average, a new employee can assemble $N(t)$ components per day after $t$ days of on-the-job training, as given by $N(t) = \frac{50t}{t+4}; t \geq 0$

Sketch a graph of $N$, $0 \leq t \leq 100$, including any vertical or horizontal asymptotes. What does $N(t)$ approach as $t$ increases without bound?

Vertical asymptote: None for $t \geq 0$.

Horizontal asymptote: $N(t) = \frac{50t}{t+4}$ The degree of $p(x) = \text{degree of } q(x)$ so $y = \frac{50}{1} = 50$

So $y = 50$ is a horizontal asymptote.

$N(t)$ approaches 50 as $t$ increases without bound. It appears that 50 components per day would be the upper limit that an employee would be expected to assemble.
Section 2.5 - Exponential Functions

Objectives:

- The student will be able to graph and identify the properties of exponential functions.
- The student will be able to graph and identify the properties of base e exponential functions.
- The student will be able to apply base e exponential functions, including growth and decay applications.
- The student will be able to solve compound interest problems.

Exponential Function

The equation \( f(x) = b^x, \quad b \neq 1 \) defines an exponential function for each different constant \( b \), called the base. The domain of \( f \) is the set of all real numbers, while the range of \( f \) is the set of all positive real numbers.

Riddle: Here is a problem related to exponential functions:
Suppose you received a penny on the first day of December, two pennies on the second day of December, four pennies on the third day, eight pennies on the fourth day and so on. How many pennies would you receive on December 31 if this pattern continues? Would you rather take this amount of money or receive a lump sum payment of $10,000,000?
Solution:

Complete the table:

<table>
<thead>
<tr>
<th>Day</th>
<th>No. pennies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>32</td>
</tr>
<tr>
<td>7</td>
<td>64</td>
</tr>
</tbody>
</table>

Now, if this pattern continued, how many pennies would you have on Dec. 31?

Your answer should be $2^{30}$ (two raised to the thirtieth power). The exponent on two is one less than the day of the month. See the preceding slide.

What is $2^{30}$? 1,073,741,824 pennies!!! Move the decimal point two places to the left to find the amount in dollars. You should get: $10,737,418.24$

This last example shows how an exponential function grows extremely rapidly. In this case, the exponential function $f(x) = 2^x$ is used to model this problem.
Graph of $f(x) = 2^x$

- Use a table to graph the exponential function above. Note: $x$ is a real number and can be replaced with numbers such as $\sqrt{2}$ as well as other irrational numbers. We will use integer values for $x$ in the table:

**Table of values**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$2^{-4} = 1/2^4 = 1/16$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$2^{-3} = 1/8$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$2^{-2} = 1/4$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$2^{-1} = 1/2$</td>
</tr>
<tr>
<td>$0$</td>
<td>$2^0 = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$2^1 = 2$</td>
</tr>
<tr>
<td>$2$</td>
<td>$2^2 = 4$</td>
</tr>
</tbody>
</table>

**Basic Properties of the Graph of $y = f(x) = b^x$, $b > 0$ and $b \neq 1$**

- All graphs will pass through $(0,1)$ (y intercept)
- All graphs are continuous curves, with no holes of jumps.
- The x axis is a horizontal asymptote.
- If $b > 1$, then $b^x$ increases as $x$ increases.
- If $0 < b < 1$, then $b^x$ decreases as $x$ increases.
Graph of \( f(x) = 2^{-x} = \frac{1}{2^x} \)

Using a table of values, you will obtain the following graph. The graphs of \( f(x) = b^x \) and \( f(x) = b^{-x} \) will be reflections of each other about the \( y \)-axis, in general.

![Graph of \( y = 2^{(-x)} \)](image)

- Graph of \( y = 2^{(-x)} \) approaches the positive \( x \)-axis as \( x \) gets large
- Graph of \( y = 2^{(-x)} \) passes through \((0,1)\)

### Properties of Exponential Functions

For \( a \) and \( b \) positive, \( a \neq 1, b \neq 1 \), and \( x \) and \( y \) real,

1. Exponent laws:
   
   \[
   a^x a^y = a^{x+y} \quad \frac{a^x}{a^y} = a^{x-y} \\
   (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}
   \]

2. \( a^x = a^y \) if and only if \( x = y \)

3. For \( x \neq 0 \), \( a^x = b^x \) if and only if \( a = b \).
Base e Exponential Functions

- Of all the possible bases $b$ we can use for the exponential function $y = b^x$, probably the most useful one is the exponential function with base $e$.
- The base $e$ is an irrational number, and, like $\pi$, cannot be represented exactly by any finite decimal fraction.
- However, $e$ can be approximated as closely as we like by evaluating the expression

$$\left(1 + \frac{1}{x}\right)^x$$

Exponential Function With Base e

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = e^x$</th>
<th>$y = e^{-x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.7182818</td>
<td>0.3678794</td>
</tr>
<tr>
<td>10</td>
<td>2.59374246</td>
<td>0.0497947</td>
</tr>
<tr>
<td>100</td>
<td>2.704813829</td>
<td>0.0497947</td>
</tr>
<tr>
<td>1000</td>
<td>2.716923932</td>
<td>0.0497947</td>
</tr>
<tr>
<td>10000</td>
<td>2.718145927</td>
<td>0.0497947</td>
</tr>
<tr>
<td>100000</td>
<td>2.718280469</td>
<td>0.0497947</td>
</tr>
</tbody>
</table>

The table to the left illustrates what happens to the expression

$$\left(1 + \frac{1}{x}\right)^x$$

as $x$ gets increasingly larger. As we can see from the table, the values approach a number whose approximation is 2.718

$y = e^x$ and $y = e^{-x}$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$
Relative Growth Rates

- Functions of the form $y = ce^{kt}$, where $c$ and $k$ are constants and the independent variable $t$ represents time, are often used to model population growth and radioactive decay.

- Note that if $t = 0$, then $y = c$. So, the constant $c$ represents the initial population (or initial amount.)

- The constant $k$ is called the relative growth rate. When $k > 0$ then growth if $k < 0$ then decay. If the relative growth rate is $k = 0.02$, then at any time $t$, the population is growing at a rate of $0.02y$ persons (2% of the population) per year.

- We say that population is growing continuously at relative growth rate $k$ to mean that the population $y$ is given by the model $y = ce^{kt}$.

Growth and Decay Applications: Atmospheric Pressure

Example: The atmospheric pressure $p$ decreases with increasing height. The pressure is related to the number of kilometers $h$ above the sea level by the formula:

- Find the pressure at sea level ($h = 0$)
- Find the pressure at a height of 7 kilometers

Solution: Find the pressure at sea level ($h = 0$)

Find the pressure at a height of 7 kilometers

$$P(0) = 760e^0 = 760$$

$$P(7) = 760e^{-0.145(7)} = 275.43$$
Depreciation of a Machine
A machine is initially worth \( V_0 \) dollars but loses 10% of its value each year. Its value after \( t \) years is given by the formula \( V(t) = V_0 (0.9)^t \)

Find the value after 8 years of a machine whose initial value is $30,000.

Solution:
\[
V(8) = 30000(0.9^8) = $12,914
\]

Compound Interest
- The compound interest formula is \( A = P \left(1 + \frac{r}{n}\right)^{nt} \)
- Here, \( A \) is the future value of the investment, \( P \) is the initial amount (principal or future value), \( r \) is the annual interest rate as a decimal, \( n \) represents the number of compounding periods per year, and \( t \) is the number of years

Compound Interest Problem
Find the amount to which $1500 will grow if deposited in a bank at 5.75% interest compounded quarterly for 5 years

Solution: Use the compound interest formula: 
\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]
Substitute \( P = 1500 \), \( r = 0.0575 \), \( n = 4 \) and \( t = 5 \) to obtain
\[
A = 1500 \left(1 + \frac{0.0575}{4}\right)^{4(5)}
\]
\[
= $1995.55
\]
Section 2.6 - Logarithmic Functions

Objectives:

- The student will be able to use and apply inverse functions.
- The student will be able to use and apply logarithmic functions and properties of logarithmic functions.
- The student will be able to evaluate logarithms.
- The student will be able to solve applications involving logarithms.

Logarithmic Functions

- In this section, another type of function will be studied called the logarithmic function. There is a close connection between a logarithmic function and an exponential function. We will see that the logarithmic function and exponential functions are inverse functions. We will study the concept of inverse functions as a prerequisite for our study of logarithmic function.

One to One Functions

We wish to define an inverse of a function. Before we do so, it is necessary to discuss the topic of one to one functions.

First of all, only certain functions are one to one.

**Definition:** A function $f$ is said to be **one-to-one** if each range value corresponds to exactly one domain value.
Graph of One to One Function

This is the graph of a one-to-one function. Notice that if we choose two different $x$ values, the corresponding $y$ values are different. Here, we see that if $x = 0$, then $y = 1$, and if $x = 1$, then $y$ is about 2.8. Now, choose any other pair of $x$ values. Do you see that the corresponding $y$ values will always be different?

Horizontal Line Test

- Recall that for an equation to be a function, its graph must pass the vertical line test. That is, a vertical line that sweeps across the graph of a function from left to right will intersect the graph only once at each $x$ value.
- There is a similar geometric test to determine if a function is one to one. It is called the horizontal line test. Any horizontal line drawn through the graph of a one to one function will cross the graph only once. If a horizontal line crosses a graph more than once, then the function that is graphed is not one to one.

Definition of Inverse Function

- If $f$ is a one-to-one function, then the inverse of $f$ is the function formed by interchanging the independent and dependent variable for $f$. Thus, if $(a, b)$ is a point on the graph of $f$, then $(b, a)$ is a point on the graph of the inverse of $f$.
- Note: If a function is not one-to-one (fails the horizontal line test) then $f$ does not have an inverse.
Logarithmic Functions
- The logarithmic function with base two is defined to be the inverse of the one to one exponential function $y = 2^x$
- Notice that the exponential $y = 2^x$ function is one to one and therefore has an inverse.

Inverse of an Exponential Function
- Start with $y = 2^x$
- Now, interchange $x$ and $y$ coordinates: $x = 2^y$
- There are no algebraic techniques that can be used to solve for $y$, so we simply call this function $y$ the logarithmic function with base 2. The definition of this new function is: $\log_2 x = y$ iff $x = 2^y$

Logarithmic Function
The inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$,
$$y = \log_b x$$ is equivalent to $x = b^y$
The **log to the base** \( b \) **of** \( x \) **is** the exponent to which \( b \) must be raised to obtain \( x \). The **domain** of the logarithmic function is the set of all positive real numbers and the **range** of the logarithmic function is the set of all real numbers.

**Graph, Domain, Range of Logarithmic Functions**

- The domain of the logarithmic function \( y = \log_b x \) is the same as the range of the exponential function \( y = b^x \). Why?
- The range of the logarithmic function is the same as the domain of the exponential function (Again, why?)
- Another fact: If one graphs any one to one function and its inverse on the same grid, the two graphs will always be symmetric with respect to the line \( y = x \).
Logarithmic-Exponential Conversions

Study the examples below. You should be able to convert a logarithmic into an exponential expression and vice versa.

1. \( \log_4(16) = x \rightarrow 4^x = 16 \rightarrow x = 2 \)

2. \( \log_3\left(\frac{1}{27}\right) = \log_3\left(\frac{1}{3^3}\right) = \log_3(3^{-3}) = -3 \)

3. \( 125 = 5^3 \rightarrow \log_5(125) = 3 \)

4. \( \sqrt{81} = 9 \rightarrow 81^{\frac{1}{2}} = 9 \rightarrow \log_{81}(9) = \frac{1}{2} \)

Solving Equations

Using the definition of a logarithm, you can solve equations involving logarithms. Examples:

\[
\log_b(1000) = 3 \rightarrow b^3 = 1000 \rightarrow b^3 = 10^3 \rightarrow b = 10
\]

\[
\log_6(x) = 5 \rightarrow 6^5 = x \rightarrow 7776 = x
\]

In each of the above, we converted from log form to exponential form and solved the resulting equation.
Properties of Logarithms
If \( b, M, \) and \( N \) are positive real numbers, \( b \neq 1, \) and \( p \) and \( x \) are real numbers, then

1. \( \log_b(1) = 0 \)
2. \( \log_b(b) = 1 \)
3. \( \log_b(b^x) = x \)
4. \( b^{\log_b x} = x \)
5. \( \log_b MN = \log_b M + \log_b N \)
6. \( \log_b \frac{M}{N} = \log_b M - \log_b N \)
7. \( \log_b M^p = p \log_b M \)
8. \( \log_b M = \log_b N \) iff \( M = N \)

Example: Solve for \( x \)
\( \log_4 (x + 6) + \log_4 (x - 6) = 3 \)

Solution:

Solve for \( x \): \( \log_4 (x + 6) + \log_4 (x - 6) = 3 \rightarrow \)
Product rule \( \log_4 (x + 6)(x - 6) = 3 \rightarrow \)
Special product \( \log_4 (x^2 - 36) = 3 \rightarrow \)
Definition of log \( 4^3 = x^2 - 36 \rightarrow \)
\( 64 = x^2 - 36 \rightarrow \)
\( 100 = x^2 \rightarrow \)
\( x \) can be +10 only \( \pm 10 = x \rightarrow \)
Why? Because you can only take logs of positive values

\( x = 10 \)
Example: Solve
\[ \log \pi + \log \left(10,000\pi\right) = x \]

Solution:

Solve:
\[ \log \pi + \log \left(10,000\pi\right) = x \]

Quotient rule
\[ \log \frac{\pi}{10,000\pi} = x \]

Simplify
(divide out common factor \(\pi\))
\[ \log \left(\frac{1}{10,000}\right) = x \]

Rewrite
\[ \log_{10} \left[ 10^{-4} \right] = x \]
\[ -4 = x \]

Common Logs and Natural Logs

- **Common log**
  \[ \log x = \log_{10} x \]
  
  If no base is indicated, the logarithm is assumed to be base 10.

- **Natural log**
  \[ \ln(x) = \log_e x \]
  
  \[ e \approx 2.7181828 \]
Solve for $x$. Obtain the exact solution of this equation in terms of $e$ (2.71828...)

$$\ln (x + 1) - \ln x = 1$$

Solution:
Application

How long will it take money to double if compounded monthly at 4% interest?

Compound interest formula

\[ A = P \left(1 + \frac{r}{m}\right)^{mt}\]

Replace \( A \) by 2\( P \) (double the amount)

\[ 2P = P \left(1 + \frac{0.04}{12}\right)^{12t} \]

Substitute values for \( r \) and \( m \)

\[ 2 = (1.003333...)^{12t} \]

Divide both sides by \( P \)

\[ \ln 2 = \ln \left(1.003333...\right)^{12t} \]

Take \( \ln \) of both sides

\[ \ln 2 = 12t \ln(1.003333...) \]

Property of logarithms

Solve for \( t \) and evaluate expression

\[ t = \frac{\ln 2}{12 \ln(1.003333...) \approx 17.36} \]

Logarithmic Regression

Among increasing functions, the logarithmic functions with bases \( b > 1 \) increase much more slowly for large values of \( x \) than either exponential or polynomial functions. When a visual inspection of the plot of a data set indicates a slowly increasing function, a logarithmic function often provides a good model. We use logarithmic regression on a graphing calculator to find the function of the form \( y = a + b \ln(x) \) that best fits the data.
Example of Logarithmic Regression
A cordless screwdriver is sold through a national chain of discount stores. A marketing company established the following price-demand table, where $x$ is the number of screwdrivers people are willing to buy each month at a price of $p$ dollars per screwdriver.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$p = D(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>91</td>
</tr>
<tr>
<td>2,000</td>
<td>73</td>
</tr>
<tr>
<td>3,000</td>
<td>64</td>
</tr>
<tr>
<td>4,000</td>
<td>56</td>
</tr>
<tr>
<td>5,000</td>
<td>53</td>
</tr>
</tbody>
</table>

Solution:

To find the logarithmic regression equation, enter the data into lists on your calculator, as shown below.

Then choose **LnReg** from the statistics menu. This means that the regression equation is $y = 256.4659 - 24.038 \ln x$
Chapter 2 Review: Important Terms, Symbols, Concepts

2.1 Functions

- **Point-by-point plotting** may be used to **sketch the graph** of an equation in two variables: plot enough points from its solution set in a rectangular coordinate system so that the total graph is apparent and then connect these points with a smooth curve.

- A **function** is a correspondence between two sets of elements such that to each element in the first set there corresponds one and only one element in the second set. The first set is called the **domain** and the second set is called the **range**.

- If \( x \) represents the elements in the domain of a function, then \( x \) is the **independent variable** or **input**. If \( y \) represents the elements in the range, then \( y \) is the **dependent variable** or **output**.

- If in an equation in two variables we get exactly one output for each input, then the equation specifies a function. The **graph** of such a function is just the graph of the equation. If we get more than one output for a given input, then the equation does not specify a function.

- The **vertical line** test can be used to determine whether or not an equation in two variables specifies a function.

- The functions specified by equations of the form \( y = mx + b \), where \( m \) is not equal to 0, are called **linear**.
functions. Functions specified by equations of the form \( y = b \) are called constant functions.

- If a function is specified by an equation and the domain is not indicated, we agree to assume that the domain is the set of all inputs that produce outputs that are real numbers.
- The symbol \( f(x) \) represents the element in the range of \( f \) that corresponds to the element \( x \) of the domain.
- Break-even and profit-loss analysis uses a cost function \( C \) and a revenue function \( R \) to determine when a company will have a loss \( (R < C) \), break even \( (R = C) \) or a profit \( (R > C) \).

### 2.2 Elementary Functions: Graphs and Transformations

- The six basic elementary functions are the identity function, the square and cube functions, the square root and cube root functions and the absolute value function.
- Performing an operation on a function produces a transformation of the graph of the function. The basic graph transformations are: vertical and horizontal translations (shifts), reflection in the x-axis, and vertical stretches and shrinks.
- A piecewise-defined function is a function whose definition involves more than one formula.

### 2.3 Quadratic Functions

- If \( a, b, \) and \( c \) are real numbers with \( a \) not equal to 0, then the function \( f(x) = ax^2 + bx + c \) is a quadratic function in standard form, and its graph is a parabola.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad b^2 - 4ac \geq 0
\]

- The quadratic formula can be used to find the \( x \) intercepts.
• Completing the square in the standard form of a quadratic functions produces the **vertex form** \( f(x) = a(x - h)^2 + k \)
• From the vertex form of a quadratic function, we can read off the vertex, axis of symmetry, maximum or minimum, and range, and sketch the graph.
• If a revenue function \( R(x) \) and a cost function \( C(x) \) intersect at a point \((x_0, y_0)\), then both this point and its coordinate \(x_0\) are referred to as **break-even points**.
• **Quadratic regression** on a graphing calculator produces the function of the form \( y = ax^2 + bx + c \) that best fits a data set.

### 2.4 Polynomial and Rational Functions

• A **polynomial function** is a function that can be written in the form \( f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \)
  
  \( n \) is the **degree**, \( a_n \neq 0 \) is the **leading coefficient**. The **domain** is the set of all real numbers.

  • The graph of a polynomial function of degree \( n \) can intersect the \( x \) axis at most \( n \) times. An \( x \) intercept is also called a **zero** or **root**.

  • The graph of a polynomial function has no sharp corners and is continuous, that is, it has no holes or breaks.

  • Polynomial regression produces a polynomial of specified degree that best fits a data set.

• A rational function is any function that can be written in the form

\[
f(x) = \frac{n(x)}{d(x)} \quad d(x) \neq 0
\]

where \( n(x) \) and \( d(x) \) are polynomials.

The **domain** is the set of all real numbers such that \( d(x) \neq 0 \).
A rational function can have vertical asymptotes [but not more than the degree of the denominator $d(x)$] and at most one horizontal asymptote.

2.5 Exponential Functions

- An **exponential function** is a function of the form $f(x) = b^x$, where $b \neq 1$ is a positive constant called the **base**. The **domain** of $f$ is the set of all real numbers and the **range** is the set of positive real numbers.
- The graph of an exponential function is continuous, passes through (0,1), and has the $x$ axis as a horizontal asymptote.
- Exponential functions obey the familiar laws of exponents and satisfy additional properties.
- The base that is used most frequently in mathematics is the irrational number $e \approx 2.7183$.
- Exponential functions can be used to model population growth and radioactive decay.
- Exponential regression on a graphing calculator produces the function of the form $y = ab^x$ that best fits a data set.
- Exponential functions are used in computations of **compound interest**:

$$A = P \left(1 + \frac{r}{m}\right)^{mt}$$

and

$$A = Pe^{rt}$$

2.6 Logarithmic Functions

- A function is said to be **one-to-one** if each range value corresponds to exactly one domain value.
- The **inverse** of a one to one function $f$ is the function formed by interchanging the independent and dependent variables of $f$. That is, $(a,b)$ is a point on the graph of $f$ if and only if
$(b,a)$ is a point on the graph of the inverse of $f$. A function that is not one to one does not have an inverse.

- The inverse of the exponential function with base $b$ is called the **logarithmic function with base $b$**, denoted $y = \log_b x$. The **domain** of $\log_b x$ is the set of all positive real numbers and the **range** of is the set of all real numbers.

- Because $y = \log_b x$ is the inverse of the function $y = b^x$, $y = \log_b x$ is equivalent to $x = b^y$.

- Properties of logarithmic functions can be obtained from corresponding properties of exponential functions.

- Logarithms to base $10$ are called **common logarithms**, denoted by $\log x$. Logarithms to base $e$ are called **natural logarithms**, denoted by $\ln x$.

- Logarithms can be used to find an investment’s **doubling time** - the length of time it takes for the value of an investment to double.

- **Logarithmic regression** on a graphing calculator produces the function of the form $y = a + b \ln x$ that best fits a data set.